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Existence of smooth solutions to a one-dimensional nonlinear degenerate variational wave equation

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ABSTRACT

This paper is focused on a one-dimensional nonlinear variational wave equation which is the Euler–Lagrange equations of a variational principle arising in the theory of nematic liquid crystals and a few other physical contexts. We establish the global existence of smooth solutions to its degenerate initial–boundary value problem under relaxed conditions on the initial–boundary data. Moreover, we show that the solution is uniformly $C^{1,\alpha}$ continuous up to the degenerate boundary and the degenerate curve is $C^{1,\alpha}$ continuous for $\alpha \in (0, \frac{1}{2})$.

1. Introduction

We are interested in a variational principle whose action is a quadratic function of the derivatives of the field with coefficients depending on the field and the independent variables

$$\delta \int A^{ij}_{\mu\nu}(\mathbf{x}, u)\frac{\partial u^{\mu}}{\partial x_i}\frac{\partial u^{\nu}}{\partial x_j}\mathrm{d}\mathbf{x} = 0, \tag{1.1}$$

where the summation convention is employed, see [2,19,20] for the background information. Here, $\mathbf{x} \in \mathbb{R}^{d+1}$ are the space–time independent variables and $\mathbf{u} : \mathbb{R}^{d+1} \to \mathbb{R}^n$ are the dependent variables, the coefficients $A^{ij}_{\mu\nu} : \mathbb{R}^{d+1} \times \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Consider $n = 1$ and $d = 1$, then the Euler–Lagrange equation of (1.1) is

$$(A^{11}u_t + A^{12}u_x)_t + (A^{21}u_t + A^{22}u_x)_x = \frac{1}{2}\left(\frac{\partial A^{11}}{\partial u}u_t^2 + \frac{\partial(A^{12} + A^{21})}{\partial u}u_t u_x + \frac{\partial A^{22}}{\partial u}u_x^2\right). \tag{1.2}$$

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The main motivation for studying (1.1) and (1.2) comes from the theory of nematic liquid crystals. In the regime in which inertia effects dominate viscosity, Saxton [28] modeled the propagation of the orientation waves in the director field of a nematic liquid crystal by the least action principle

$$\delta \int \left(\frac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) \right) \mathrm{d}\mathbf{x} \mathrm{d}t = 0, \qquad \mathbf{n} \cdot \mathbf{n} = 1, \tag{1.3}$$

where $\mathbf{n}(\mathbf{x}, t)$ is the director field and $W(\mathbf{n}, \nabla \mathbf{n})$ is the well-known Oseen–Frank potential energy density,

$$W(\mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} k_1 (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} k_2 (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + \frac{1}{2} k_3 |\mathbf{n} \times (\nabla \times \mathbf{n})|^2.$$

Here k_1, k_2 and k_3 are the splay, twist and bend elastic constants of the liquid crystal, respectively. In [19], Hunter and Saxton considered the planar deformations of the director field $\mathbf{n}$ that depend only on a single space variable x with $\mathbf{n} = (\cos u(t,x), \sin u(t,x), 0)$ and derived the Euler–Lagrange equation of (1.3) given by

$$u_{tt} - c(u)[c(u)u_x]_x = 0 \tag{1.4}$$

with

$$c^2(u) = k_1 \sin^2 u + k_3 \cos^2 u.$$

See [19,28,40] for the details on the derivation and physical background of the above equation. Generally, the elastic constants k_1 and k_3 are positive and then the wave speed $c(\cdot)$ is a strictly positive function. However in some cases, see e.g. [1,10,27], the elastic constant k_1 or k_3 may be negative which implies that the wave speed $c(\cdot)$ can be zero. For another application, if $c(u) = u$, then (1.4) is reduced to the second sound equation in one space dimension [21].

Under the assumption that the wave speed $c(\cdot)$ is a positive function, the nonlinear variational wave equation (1.4) has been widely explored. Glassey et al. [11] shown that smooth solutions of (1.4) may develop cusp-type singularities in finite time due to the nonlinear nature. There are many authors considering the global existence of two natural distinct classes of weak solutions (dissipative and conservative) to its initial data problem. In a series of papers [33–37], Zhang and Zheng have studied carefully the global existence of dissipative solutions for (1.4) and its asymptotic models by using the Young measure theory. A different approach to construct a global dissipative solution was taken in [6]. In [7], Bressan and Zheng introduced the method of energy-dependent coordinates and established the global existence of conservative solutions to its Cauchy problem for initial data of finite energy. The uniqueness of conservative solutions has been proved recently by Bressan et al. [5]. Holden and Raynaud [13] provided a detailed construction of a global semigroup for conservative solutions of (1.4). A Lipschitz continuous metric to its conservative solutions has been constructed recently by Bressan and Chen [4]. For more related results about (1.4), see, among others, [8,12,15] and references therein. We also refer the reader to Refs. [3,14,38,39] for a discussion of the global existence of conservative solutions to the one-dimensional nonlinear variational wave systems.

Until recently, to the best of our knowledge, studies on the degenerate hyperbolic problems for the nonlinear variational wave equations are still very limited. We expect this work can help unveiling the structure of solutions near a degenerate curve for variational wave equations (1.2) and (1.4). Due to the complexity, in this paper we consider the following simplified equation of (1.2)

$$u_{tt} - [c^2(u,x)u_x]_x = -c(u,x)c_u(u,x)u_x^2 \tag{1.5}$$

where the smooth function $c(u,x)$ is independent of time t. We note that this equation includes (1.4) as a special case. Under the positivity assumption on c, the global existence of conservative weak solutions to