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An elliptic free boundary arising from the jump of conductivity

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ABSTRACT

In this paper we consider a quasilinear elliptic PDE, $\mathrm{div}(A(x,u)\nabla u) = 0$, where the underlying physical problem gives rise to a jump for the conductivity $A(x,u)$, across a level surface for u. Our analysis concerns Lipschitz regularity for the solution u, and the regularity of the level surfaces, where $A(x,u)$ has a jump and the solution u does not degenerate.

In proving Lipschitz regularity of solutions, we introduce a new and unexpected type of ACF-monotonicity formula with two different operators, that might be of independent interest, and surely can be applied in other related situations. The proof of the monotonicity formula is done through careful computations, and (as a byproduct) a slight generalization to a specific type of variable matrix-valued conductivity is presented.

1. Introduction

1.1. The model equation

In this paper, we consider weak solutions u of

$$\mathrm{div}(A(x,u)\nabla u) = 0 \quad \text{in } \Omega, \tag{L}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and $A : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined by

$$A(x,s) := a_-(x) + (a_+(x) - a_-(x))H(s) = a_-(x)(1 - H(s)) + a_+(x)H(s), \tag{1.1}$$

where H is the Heaviside function and $a_\pm \in C(\mathbb{R}^n)$ satisfying the following structure conditions:

- there is a $\lambda > 0$ such that

$$\lambda \leq \min\{a_-(x), a_+(x)\}, \qquad \max\{a_-(x), a_+(x)\} \leq \frac{1}{\lambda}, \quad \forall x \in \mathbb{R}^n; \tag{1.2}$$

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- there is a modulus of continuity ω such that

$$\max\{|a_-(x) - a_-(y)|, |a_+(x) - a_+(y)|\} \leq \omega(|x-y|), \quad \forall x, y \in \mathbb{R}^n. \tag{1.3}$$

We call u a weak solution of (L), if $u \in W^{1,2}_{loc}(\Omega) \cap L^2(\Omega)$ satisfies

$$\int A(x,u)\nabla u \nabla \phi = 0, \quad \forall \phi \in W^{1,2}_0(\Omega).$$

1.2. Applications

Heat or electric conduction through certain materials are usually, due to complex properties of the materials, very hard to model. For instance, mathematical modeling of composites, consisting of materials with different conductivity properties, is one such problem which has been subject for intense (mathematical) studies. Applications of such models can be found in problems related to transmissions [8], and inverse and discontinuous conductivity [1], as well as other related problems [9], where there is a jump in the conductivity.

The conductivity problem becomes substantially complicated when the materials also undergo a phase transition, which can cause abrupt changes in the conductivity, due to a threshold of the heat or electric current. The discontinuity in the conduction that arises from a structural phase change in crystalline materials has been considered in applied literature specially in relation to transport in solids, such as nanowires. Relevant discussions in applied literature can be found in [7,19]. The mathematical problem of jump in conductivity across level surfaces has also been considered recently in [5,6] and [4].

The simplest model of a material-dependent conductivity can be written as $A(x,u)\nabla u$, where now $A(x,u)$ has a discontinuity in the u-variable, which represents the heat, or electric charge. In this paper, we have chosen a simple elliptic model, where the model equation is written as $A(x,u) = a_+(x)\chi_{\{u>0\}} + a_-(x)\chi_{\{u\leq 0\}}$.

It should be remarked that heat conduction in certain materials, which also undergo a phase transition, may cause a change in conductivity only at the phase, in terms of latent heat on the boundary between two-phases. Typical examples are melting ice or flame propagation. The mathematical study of such problems, usually entitled Bernoulli free boundaries, has been carried out in a large scale in the last few decades, e.g., [2,3,12] and the references therein. Our problem, although qualitatively different, carries features reminiscent of those of Bernoulli type problem for the latent heat.

1.3. Methodology and approach

As appearing in (1.1), the jump of $A(x,u)$ in the second argument across $\{u = 0\}$ naturally induces a free boundary condition, which can be formally represented as

$$a_+(x)u^+_\nu = a_-(x)u^-_\nu,$$

where ν points towards level sets where u increases.

Assuming the continuity (1.3) of our coefficients, it is not hard to see that our limiting equation after blowing up the solution at a point z is of the form

$$\mathrm{div}((a_-(z) + (a_+(z) - a_-(z))H(v))\nabla v) = 0,$$

so that the function w, defined by

$$w(x) := a_+(z)v^+(x) - a_-(z)v^-(x),$$