Nonlinear Analysis 161 (2017) 30–43

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the construction of gradient Ricci soliton warped product

F.E.S. Feitosa, A.A. Freitas Filho, J.N.V. Gomes*

Departamento de Matemática-ICE-UFAM, 69080-900, Manaus-AM, Brazil

ARTICLE INFO

Article history:
Received 3 March 2017
Accepted 26 May 2017
Communicated by Enzo Mitidieri

MSC 2010:
primary 53C24
53C25
secondary 53C15
53C21

Keywords:
Ricci soliton
Warped product
Scalar curvature
Rigidity results

ABSTRACT

In this paper we show that an expanding or steady gradient Ricci soliton warped product $B^n \times_f F^m$, $m > 1$, whose warping function f reaches both maximum and minimum must be a Riemannian product. Moreover, we present a necessary and sufficient condition for constructing a gradient Ricci soliton warped product. As an application, we present a class of expanding Ricci soliton warped product having as a fiber an Einstein manifold with non-positive scalar curvature. We also discuss some obstructions to this construction, especially in the case when the base of the warped product is compact.

1. Introduction

The study of warped products have been of great interest throughout the recent years. This concept was first introduced by Bishop and O'Neill as they succeeded to give examples of complete Riemannian manifolds with negative sectional curvature [5]. Given two Riemannian manifolds (B, g_B) and (F, g_F) as well as a positive smooth function f on B, we define on the product manifold $B \times F$ the metric

$$g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \tag{1.1}$$

where π and σ are the natural projections on B and F, respectively. Under these conditions the product manifold is said to be the *warped product* of B and F; it is denoted $M = B \times_f F$ and the function f is called the *warping function*. Notice that when f is constant M is just the usual Riemannian product. Albeit the class of warped products with non-constant warping functions provides a rich class of examples in

* Corresponding author.
E-mail addresses: feitosaufam@gmail.com (F.E.S. Feitosa), freitas.airtonfilho@hotmail.com (A.A. Freitas Filho), jnvgomes@pq.cnpq.br (J.N.V. Gomes).
URL: http://www.ufam.edu.br (J.N.V. Gomes).

http://dx.doi.org/10.1016/j.na.2017.05.013

Riemannian geometry, it was shown by Kim-Kim [22] that there does not exist a compact Einstein warped product with non-constant warping function if the scalar curvature is non-positive. Moreover, they observed that a necessary condition for a warped product be an Einstein manifold is its base be a quasi-Einstein metric, i.e., a Riemannian manifold whose modified Bakry–Emery Ricci tensor is a constant multiple of the metric tensor. One should point out that some examples of expanding quasi-Einstein manifolds having as a fiber an arbitrary Einstein manifold as well as steady quasi-Einstein manifolds with fiber of non-negative scalar curvature were constructed in Besse [4]. More recently, Barros–Batista–Ribeiro [2] provided some volume estimates for Einstein warped products similar to a classical result due to Calabi [8] and Yau [26] for complete Riemannian manifolds with non-negative Ricci curvature. For this, they made use of the approach of quasi-Einstein manifolds. In particular, they also presented an obstruction for the existence of such a class of manifolds. We would like to mention here the work of He–Petersen–Wylie [19] concerning warped product Einstein manifolds. Being an extension of the work of Case–Shu–Wei [11] and some earlier work of Kim-Kim [22], the upshot of [19] is that the base may have non-empty boundary.

A natural generalization of the Einstein manifolds is the Ricci solitons. This concept was introduced by Hamilton [16] in early 80s. We recall that a Ricci soliton is a complete Riemannian manifold M endowed with a metric g, a vector field $X \in \mathfrak{X}(M)$ and a constant λ satisfying the equation

$$Ric + \frac{1}{2}\mathscr{L}_X g = \lambda g. \tag{1.2}$$

We shall refer to this equation as the fundamental equation. A Ricci soliton is called *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. When $X = \nabla\psi$ for some smooth function ψ on M, we write $(M, g, \nabla\psi, \lambda)$ for the gradient Ricci soliton with potential function ψ. In this case, the fundamental equation can be rewritten as

$$Ric + \nabla^2\psi = \lambda g, \tag{1.3}$$

where $\nabla^2\psi$ denotes the Hessian of ψ. For more details see [9,16]. It has been known since the early 90s that a compact gradient steady or expanding Ricci soliton is necessarily an Einstein manifold [18,20]. In [24], Petersen and Wylie used a theorem due to Brinkmann [6] to show that any surface gradient Ricci soliton is a warped product. It is also known that Robert Bryant (see [7,12]) constructed a steady Ricci soliton as the warped product $(0, +\infty) \times_f \mathbb{S}^m$, $m > 1$, with a radial warping function f. Since this latter function is not limited we reach the following natural question: *Under which conditions a warped product with a limited warping function is a Ricci soliton?* Our first theorem gives a partial answer to this question.

Theorem 1. *Let $M = B^n \times_f F^m$ be a warped product and φ a smooth function on B so that $(M, g, \nabla\tilde{\varphi}, \lambda)$ be an expanding or steady gradient Ricci soliton. Assume that its fiber F^m is of dimension at least two and that its warping function f reaches both maximum and minimum. Then M must be a Riemannian product.*

This latter theorem is motivated by the ideas of [22] which concern compact Einstein warped product spaces with non-positive scalar curvature. We point out that Theorem 1 is a natural generalization of the Einstein case to the Ricci soliton case without the compactness condition on the product that was taken in [22]. Incidentally, an interesting fact emerges when we study Ricci solitons that are realized as a warped product. Indeed, their bases satisfy Eq. (1.4). This is a generalization of the Einstein metrics, which contains quasi-Einstein metrics (see p. 6).

The next result establishes a compactness criterion of shrinking gradient Ricci soliton warped product under the condition that the base is compact.

Theorem 2. *Let $M = B^n \times_f F^m$ be a warped product and φ a smooth function on B so that $(M, g, \nabla\tilde{\varphi}, \lambda)$ be a shrinking gradient Ricci soliton with compact base and fiber with dimension at least two. Then M must be a compact manifold.*