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The initial–boundary value problem for general non-local scalar conservation laws in one space dimension

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ABSTRACT

We prove a global well-posedness result for a class of weak entropy solutions of bounded variation (BV) of scalar conservation laws with non-local flux on bounded domains, under suitable regularity assumptions on the flux function. In particular, existence is obtained by proving the convergence of an adapted Lax–Friedrichs algorithm. Lipschitz continuous dependence from initial and boundary data is derived applying Kružhkov's doubling of variable technique.

1. Introduction

Given a bounded open interval $I =]a, b[\subset \mathbb{R}$, we consider the following initial–boundary value problem

$$\partial_t \rho + \partial_x f(t, x, \rho, \rho * \eta) = 0, \qquad (t, x) \in \mathbb{R}^+ \times I, \tag{1.1a}$$
$$\rho(0, x) = \rho_0(x), \qquad x \in I, \tag{1.1b}$$
$$\rho(t, a) = \rho_a(t), \qquad t \in \mathbb{R}^+, \tag{1.1c}$$
$$\rho(t, b) = \rho_b(t), \qquad t \in \mathbb{R}^+, \tag{1.1d}$$

where $f \in \mathbf{C^2}(\mathbb{R}^+ \times \bar{I} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfies

$$f(t, x, 0, R) = 0 \quad \forall t, x, R, \tag{1.2a}$$
$$\sup_{t,x,\rho,R} |\partial_\rho f(t, x, \rho, R)| < L, \tag{1.2b}$$
$$\sup_{t,x,R} |\partial_x f(t, x, \rho, R)| < C\,|\rho|\,, \quad \sup_{t,x,R} |\partial_R f(t, x, \rho, R)| < C\,|\rho|\,, \tag{1.2c}$$
$$\sup_{t,x,R} \left|\partial^2_{xx} f(t, x, \rho, R)\right| < C\,|\rho|\,, \quad \sup_{t,x,R} \left|\partial^2_{xR} f(t, x, \rho, R)\right| < C\,|\rho|\,, \quad \sup_{t,x,R} \left|\partial^2_{RR} f(t, x, \rho, R)\right| < C\,|\rho|, \tag{1.2d}$$

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for some constants $L > 0$ and $C > 0$, and $\eta \in (\mathbf{C^1} \cap \mathbf{W}^{1,\infty})(\mathbb{R};\mathbb{R})$ is a convolution kernel (not necessarily with compact support) such that

$$\int_{\mathbb{R}} \eta(x)dx = 1.$$

Equations of type (1.1a) arise in several applications, and have made the object of a large literature in recent years. Space-integral terms appear for example in models for granular flows [3], sedimentation [7], supply chains [19], conveyor belts [18], weakly coupled oscillators [2], structured populations dynamics [24], or more general problems like gradient constrained equations [4]. Equations with non-local flux have been recently introduced also in traffic flow modelling to account for the reaction of drivers or pedestrians to the surrounding density of other individuals, see [8,10,11,27].

General analytical results on non-local conservation laws, proving existence and eventually uniqueness of solutions of the Cauchy problem for (1.1a), can be found in [5] for scalar equations in one space dimension, in [12] for scalar equations in several space dimensions and in [1,13,14] for multi-dimensional systems of conservation laws. Besides, specific finite volume numerical methods have been developed recently in [1,17,21]. To our knowledge, initial–boundary value problems of the form (1.1) have not been rigorously studied yet, the difficulties lying in the presence of the non-local term, which may exceed the boundaries of the space domain. Nonetheless, real applications (confined environments, networks, etc.) and numerical computations require a precise account for boundary conditions.

The scope of the present article is to propose an approach for a rigorous treatment of boundary conditions, in the case of one space-dimensional problems. The strategies we employ are inspired by classical results on scalar conservation laws with boundary conditions. In particular, we refer to [6,9,28]. Our results are based on the extension of the solution outside the domain, set to be constantly equal to the corresponding boundary condition values. In principle, other extensions may be applied, and we expect that, under suitable assumptions, they would led to the definition of corresponding well-posed problems. In this first tentative approach, being interested in testing the validity of the methodology, we have made the choice of taking the trivial extension to a constant state. Yet, this choice is reasonable in many application examples, like for example whenever the outflow is set to be equal to zero, as in the case of a red traffic light on a road. In this case, the corresponding boundary condition (equal to the maximal density) could be reproduced extending the solution constantly. On the contrary, it is far from obvious to generalize the present technique to problem in several space-dimensions with possibly non-constant boundary data.

As in the classical case, we assume that boundary conditions cannot generally be satisfied in strong sense. Therefore, we introduce the following notion of weak entropy solution for (1.1), which extends to problems with boundaries the definition of solution given in [5] for the corresponding Cauchy problem. This formulation, based on semi Kružhkov entropies [23,28], has the advantage of not using explicitly the traces of the solution at the boundaries of the domain, which turns particularly useful in the existence proof, provided in Section 2.

Definition 1. Let $\rho_0 \in \mathbf{L^\infty}(I;\mathbb{R})$ and $\rho_a, \rho_b \in \mathbf{L^\infty}(\mathbb{R}^+;\mathbb{R})$. A map $\rho \in \mathbf{L^\infty}(\mathbb{R}^+ \times I;\mathbb{R})$ is a weak entropy solution to (1.1) if for every test function $\varphi \in \mathbf{C_{c}^1}(\mathbb{R}^2;\mathbb{R}^+)$ and for every $\kappa \in \mathbb{R}$

$$\begin{aligned}
&\int_0^{+\infty}\int_a^b \Big((\rho-\kappa)^{\pm}\,\partial_t\varphi + \operatorname{sgn}(\rho-\kappa)^{\pm}\ \left(f(t,x,\rho,R(t,x)) - f(t,x,\kappa,R(t,x))\right)\partial_x\varphi \\
&\quad - \operatorname{sgn}(\rho-\kappa)^{\pm}\,\frac{d}{dx} f(t,x,\kappa,R(t,x))\,\varphi \Big)\,dx\,dt + \int_a^b (\rho_0-\kappa)^{\pm}\,\varphi(0,x)\,dx \\
&\quad + Lip(f)\int_0^{+\infty} (\rho_a(t)-\kappa)^{\pm}\varphi(t,a)\,dt + Lip(f)\int_0^{+\infty} (\rho_b(t)-\kappa)^{\pm}\varphi(t,b)\,dt \ge 0, \qquad (1.3)
\end{aligned}$$