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Existence, uniqueness and global behavior of the solutions to some nonlinear vector equations in a finite dimensional Hilbert space

Mama Abdelli[a], María Anguiano[b], Alain Haraux[c,*]

[a] *Laboratory of Analysis and Control of Partial Differential Equations, Djillali Liabes University, P. O. Box 89, Sidi Bel Abbes 22000, Algeria*
[b] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, P. O. Box 1160, 41080-Sevilla, Spain*
[c] *Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France*

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ABSTRACT

The initial value problem and global properties of solutions are studied for the vector equation: $\left(\|u'\|^l u'\right)' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0$ in a finite dimensional Hilbert space under suitable assumptions on g.

1. Introduction

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|\cdot\|$. We consider the following nonlinear equation

$$\left(\|u'\|^l u'\right)' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0, \tag{1.1}$$

where l and β are positive constants, A is a positive, symmetric linear operator on H and the derivatives are taken with respect to a variable $t \in \mathbb{R}^+$. We denote by $(\cdot,\cdot)$ the inner product in H. The operator A is coercive, which means:

$$\exists \lambda > 0, \;\; \forall u \in H, \;\; (Au, u) \geq \lambda \|u\|^2.$$

* Corresponding author.
E-mail addresses: abdelli.mama@gmail.com (M. Abdelli), anguiano@us.es (M. Anguiano), haraux@ann.jussieu.fr (A. Haraux).

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We also define

$$\forall u \in H,\ \|u\|_{D(A^{\frac{1}{2}})} := \|A^{\frac{1}{2}}u\| = (Au, u)^{\frac{1}{2}},$$

a norm equivalent to the norm in H, and for the convenience of quadratic estimates in intermediate calculations we fix some constants $\lambda, \kappa > 0$ such that

$$\forall u \in H, \quad \lambda\|u\|^2 \leq \|A^{\frac{1}{2}}u\|^2 \leq \kappa\|u\|^2.$$

We assume that $g : H \to H$ is locally Lipschitz continuous. Some additional hypotheses will be specified when needed.

When $H = \mathbb{R}$, $l = 0$ and $g(u') = c|u'|^{\alpha}u'$, existence and uniqueness of bounded global solutions are immediate and Haraux [6] studied the rate of decay of the energy of non-trivial solutions as $t \to +\infty$. In addition, he showed that all non-trivial solutions are oscillatory if $\alpha > \frac{\beta}{\beta+2}$ and non-oscillatory if $\alpha < \frac{\beta}{\beta+2}$. In the oscillatory case he established that all non-trivial solutions have the same decay rates, while in the non-oscillatory case he showed the coexistence of exactly two different decay rates, calling slow solutions those which have the lowest decay rate, and fast solutions the others.

Then, Abdelli and Haraux [1] studied for $l > 0$ the scalar ODE where $g(u') = c|u'|^{\alpha}u'$, they proved the existence and uniqueness of a global solution with initial data $(u_0, u_1) \in \mathbb{R}^2$. They used some modified energy function to estimate the rate of decay and they adapted the method introduced by Haraux [6] to study the oscillatory or non-oscillatory of non-trivial solutions. All non-trivial solutions are oscillatory if $\alpha > \frac{\beta(l+1)+l}{\beta+2}$ and non-oscillatory if $\alpha < \frac{\beta(l+1)+l}{\beta+2}$. In the non-oscillatory rate, as in the case $l = 0$, the coexistence of exactly two different decay rates was established.

In this article, we use some techniques from Abdelli and Haraux [1] to establish a global existence and uniqueness result of the solutions, and under some additional conditions on g (typically $g(s) \sim c\|s\|^{\alpha}s$), we study the asymptotic behavior as $t \to \infty$. A basic role will be played by the total energy of the solution u given by the formula

$$E(t) = \frac{l+1}{l+2}\|u'(t)\|^{l+2} + \frac{1}{\beta+2}\|A^{\frac{1}{2}}u(t)\|^{\beta+2}. \tag{1.2}$$

The plan of this paper is as follows: In Section 2 we establish some basic preliminary inequalities, and in Section 3 we prove the existence of a solution $u \in \mathcal{C}^1(\mathbb{R}^+, H)$ with $\|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H)$ for any initial data $(u_0, u_1) \in H \times H$ under relevant conditions on g and the conservation of total energy for each such solution. In Section 4 we establish the uniqueness of the solution in the same regularity class under additional conditions on g. In Section 5 we prove convergence of all solutions to 0 under more specific conditions on g and we estimate the decay rate of the energy. Finally, in Section 6, we discuss the optimality of these estimates when $g(s) = c\|s\|^{\alpha}s$ and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$; in particular, by relying on a technique introduced by Ghisi et al. [4], improving the main result of [5] we prove the existence of an open set of initial states giving rise to slow decaying solutions. In our last result, by relying on a technique introduced by Ghisi et al. [3], we prove that all non-zero solutions are either slow solutions or fast solutions.

2. Some basic inequalities

In this section, we establish some easy but powerful lemmas which generalize Lemmas 2.2 and 2.3 from [2] and will be essential for the existence and uniqueness proofs of the next section. Throughout this section, H denotes an arbitrary (not necessary finite dimensional) real Hilbert space with norm denoted by $\|\cdot\|$.