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Absolutely Minimizing Lipschitz Extensions and infinity harmonic functions on the Sierpinski gasket

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A B S T R A C T

Aim of this note is to study the infinity Laplace operator and the corresponding Absolutely Minimizing Lipschitz Extension problem on the Sierpinski gasket in the spirit of the classical construction of Kigami for the Laplacian. We introduce a notion of infinity harmonic functions on pre-fractal sets and we show that these functions solve a Lipschitz extension problem in the discrete setting. Then we prove that the limit of the infinity harmonic functions on the pre-fractal sets solves the Absolutely Minimizing Lipschitz Extension problem on the Sierpinski gasket.

1. Introduction

The theory of Absolutely Minimizing Lipschitz Extension [1,4] concerns the classical problem of extending a Lipschitz continuous function f defined on the boundary of an open set $U \subset \mathbb{R}^d$ to the interior of U without increasing the Lipschitz constant. In other words, to find a Lipschitz continuous function $u : \overline{U} \to \mathbb{R}$ such that $u = f$ on ∂U and $\mathrm{Lip}(u, U) = \mathrm{Lip}(f, \partial U)$ (Lip denotes the Lipschitz constant).

The previous problem has several solutions, all in between a maximal and a minimal one called respectively the McShane's extension and the Whitney's extension. But, among all these possible solutions, the "canonical" one is the so called Absolutely Minimizing Lipschitz Extension (AMLE in short). This function is characterized by satisfying the extension problem not only in U, but also in any open subset V of U, that is $\mathrm{Lip}(u, V) = \mathrm{Lip}(u, \partial V)$ for any open $V \subset U$. The relevance of the notion of AMLE relies in the several additional properties that this function satisfies, for example it is the unique viscosity solution

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of the Dirichlet problem for the infinity Laplace equation

$$\Delta_\infty u(x) = 0, \quad x \in U. \tag{1.1}$$

Here $\Delta_\infty u = \sum_{i,j=1}^{d} \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u$, the infinity Laplacian, is a nonlinear degenerate second order operator and a function satisfying (1.1) is said infinity harmonic.

The theory of AMLE has been extended to length spaces (see [1,3,9,10,15]), hence it applies in particular to the Sierpinski gasket $\mathcal{S}$ endowed with its geodesic distance. Prescribed a boundary data f on the vertices of the initial simplex from which $\mathcal{S}$ is obtained via iteration, there exists a unique AMLE of f to $\mathcal{S}$. Moreover this function can be characterized as in the Euclidean case by an intuitive geometric property, called Comparison with Cones.

After the seminal work of Kigami [11], a standard way to define a harmonic function on the Sierpinski gasket, and more in general for the class of post-critically finite fractals, is to consider the uniform limit of solutions of suitable scaled finite differences on the pre-fractals.

For the infinity Laplace operator this approach has been pursued in [5]. In this thesis, a graph infinity Laplacian on pre-fractal sets is defined and an algorithm to compute explicitly infinity harmonic functions is studied. By means of a constructive approach based on the previous algorithm, it is proved that the sequence of the infinity harmonic functions on the pre-fractal sets converges to a function defined on the limit fractal set. It is worth noting that the same graph infinity Laplacian is used in the Euclidean case to approximate the viscosity solution of (1.1) (see [12–14]).

Following an approach similar to [5], we aim to define an infinity harmonic function on $\mathcal{S}$ as the limit of solutions of finite difference equations on pre-fractals. We study the Lipschitz extension problem on the pre-fractal sets and we show that an appropriate notion of AMLE can be introduced in this framework. We also prove that an AMLE is a solution of the graph infinity Laplacian and it satisfies a Comparison with Cone property with respect to the path distance. The Comparison with Cone property on the pre-fractals is crucial since it allows us to show that the limit of the AMLEs on the pre-fractals is an AMLE on the Sierpinski. The convergence result allows us to define an infinity harmonic function on the Sierpinski as the limit of infinity harmonic functions on the pre-fractals and to conclude the equivalence, as in Euclidean case, between AMLE and infinity harmonic functions. Hence, besides giving a simpler proof of the convergence result in [5], we also obtain as in the Euclidean case the equivalence among the various properties which characterize the AMLE theory. We remark that the previous construction on the Sierpinski gasket can be readily extended to the class of post-critically finite fractals since it is only based on the convergence of the path distance defined on the pre-fractal sets to the path distance (intrinsic length) on the limit fractal set [2,7].

The paper is organized as follows. In Section 2 we introduce notations and definitions and we prove some preliminary properties for the graph infinity Laplacian. Section 3 is devoted to the AMLE problem on pre-fractals. In Section 4 we recall the definition of AMLE in metric spaces and we prove the convergence result for the pre-fractals invading the Sierpinski gasket. Finally, in Section 5 we describe the algorithm in [5] and we study the relation between infinity and p-harmonic function in the pre-fractals.

2. Notations and preliminary results

In this section we introduce notations and definitions and we collect some preliminary results on infinity harmonic functions on graphs.

Consider a unitary equilateral triangle V^0 of vertices $\{q_1, q_2, q_3\}$ in $\mathbb{R}^2$ and the maps $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$\psi_i(x) := q_i + \frac{1}{2}(x - q_i).$$