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Noncoercive elliptic equations with discontinuous coefficients in unbounded domains

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ABSTRACT

In this paper we study Dirichlet problems for noncoercive linear elliptic equations with discontinuous coefficients in unbounded domains. Exploiting a nonlinear approach, we achieve existence, uniqueness and regularity results.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^N$, $N > 2$. We consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(u\, E(x)) + f(x) & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$\alpha|\xi|^2 \le M(x)\,\xi \cdot \xi, \qquad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \ \forall\, \xi \in \mathbb{R}^N, \tag{2}$$

$$\mu > 0, \tag{3}$$

$E : \Omega \to \mathbb{R}^N$ is a vector field and $f : \Omega \to \mathbb{R}$ is a real function.

Guido Stampacchia proved that if Ω is bounded, $|E| \in L^N(\Omega)$, $f \in L^{\frac{2N}{N+2}}(\Omega)$ and if μ is large enough, problem (1) admits a unique weak solution u.

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Moreover, he also proved that
– if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then the solution u of (1) is in $L^{m^{**}}(\Omega)$, with

$$m^{**} = (m^*)^* = \frac{Nm}{N-2m}, \tag{4}$$

m^* being the Sobolev conjugate of m;
– if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then the solution u of (1) is in $L^\infty(\Omega)$.

Successively, Lucio Boccardo, in [1], studied the case $\mu = 0$, obtaining the same results. We point out that the main difficulty here lies in the noncoercitivity of the operator $-\mathrm{div}(M(x)\nabla u) + \mathrm{div}(u\,E(x))$ due to the presence of the second term on which no smallness assumptions are done.

In this paper we generalize these existence, uniqueness and regularity results to the case when Ω is unbounded. We explicitly observe that, since the domain is unbounded, we need to assume hypothesis (3). Nevertheless, we still have the noncoercitivity of the operator since we do not require that μ is large enough.

More precisely, we prove the unique solvability of problem (1) under the assumptions

$$|E| \in L^2(\Omega) \cap M_0^N(\Omega) \tag{5}$$

and

$$f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega), \tag{6}$$

where $M_0^N(\Omega)$ is a functional space strictly containing $L^N(\Omega)$, described in Section 2.

Furthermore, we also generalize to the case of unbounded domains the regularity results proving that
– if $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then the solution u of (1) is in $L^{m^{**}}(\Omega)$, with m^{**} given by (4);
– if $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, then the solution u of (1) is in $L^\infty(\Omega)$.

The techniques used to achieve these results issue from an idea of [1], inspired by the papers of Guido Stampacchia [23,24], and by [7,8,10], where certain nonlinear problems are treated. In [1] the author approximates the noncoercive problem by coercive nonlinear problems and then passes to the limit. Here, due to the assumption (5) on the coefficient appearing in the noncoercive term, one can pass to the limit thanks to a compactness result in $M_0^N(\Omega)$ proved in [26] (see also Lemma 2.2).

For similar problems on bounded domains we refer the reader also to [2,3,5,6,11,21,27]. Linear coercive problems on unbounded domains are studied in [15–20].

2. The spaces $M^p(\Omega)$ and $M_0^p(\Omega)$

From now on, let Ω be an unbounded subset of $\mathbb{R}^N$, $N > 2$. We start recalling the definitions and some properties of a class of spaces that were introduced for the first time in [25].

Let us give some notation. The σ-algebra of all Lebesgue measurable subsets of Ω is denoted by $\Sigma(\Omega)$. Given $O \in \Sigma(\Omega)$, $|O|$ is its Lebesgue measure, χ_O is its characteristic function, and $O(x,r)$ is the intersection $O \cap B(x,r)$ ($x \in \mathbb{R}^N$, $r \in \mathbb{R}_+$), where $B(x,r)$ is the open ball with center in x and radius r. The class of restrictions to $\bar{\Omega}$ of functions $\zeta \in C_o^\infty(\mathbb{R}^n)$ is denoted by $\mathcal{D}(\bar{\Omega})$. For $p \in [1,+\infty[$, $L^p_{loc}(\bar{\Omega})$ is the class of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^p(\Omega)$ for any $\zeta \in \mathcal{D}(\bar{\Omega})$.

For $p \in [1,+\infty[$, the space $M^p(\Omega)$ is the set of all the functions g in $L^p_{loc}(\bar{\Omega})$ such that

$$\|g\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|g\|_{L^p(\Omega(x,1))} < +\infty, \tag{7}$$

endowed with the norm defined in (7). Moreover $M_0^p(\Omega)$ denotes the subspace of $M^p(\Omega)$ made up of functions $g \in M^p(\Omega)$ such that

$$\lim_{x\to+\infty} \|g\|_{L^p(\Omega(x,1))} = 0.$$