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Radially symmetric shadow wave solutions to the system of pressureless gas dynamics in arbitrary dimensions

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ABSTRACT

Radially symmetric shadow wave solutions to the system of multidimensional pressureless gas dynamics are introduced, which allow one to capture concentration of mass. The transformation to a one-dimensional system with source terms is performed and physically meaningful boundary conditions at the origin are determined. Entropy conditions are derived and applied to single out physical (nonnegative mass) and dissipative (entropic) solutions. A complete solution to the pseudo-Riemann problem with initial data exhibiting a single delta shock on a sphere is obtained.

1. Introduction

This paper is devoted to radially symmetric nonclassical solutions to the multidimensional pressureless Euler system

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \\ \partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) &= 0 \end{aligned} \tag{1.1}$$

which consists of the convection part of the equations of conservation of mass and momentum in isentropic gas dynamics [9, Section 3.3]. System (1.1) also describes the behavior of a gas consisting of sticky particles,

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i.e. particles that stick together if they collide. This model and related models (sticky particles, adhesion particle dynamics, with or without viscosity) have a long history in cosmology, see e.g. the representative articles [25,28,29].

Following the seminal work of [6,24], the one-dimensional case has found widespread attention. It has been evident from the beginning that solutions become measure-valued in finite time even for regular initial data. Thus solution concepts have been developed that admit accommodating nonclassical solutions such as delta shocks. We list a selection of results: The notion of duality solutions has been introduced in [2]. Further existence and uniqueness results on measure valued solutions have been obtained by various methods, including the mentioned duality approach [17,18,23], particle approximations [15,22], zero viscosity limits [4], zero pressure limits [10], and optimal transport theory [8]. For numerical methods, we refer to [1,3,5]. Another approach, namely shadow waves, is due to the first author. Shadow waves were introduced for the one-dimensional pressureless Euler system, among other conservation laws, in [21], and will be the method of choice in this paper.

Concerning nonclassical solutions in the higher dimensional case, not so much seems to be known about system (1.1), although this case is important in applications. There is work on radially symmetric solutions to related systems of magnetohydrodynamics [7,14,26,27] and to the isentropic Euler equations [20]. The numerically inspired weak asymptotic method has been applied to various systems of conservation laws, in particular to system (1.1), in [11]. As general references for rotationally symmetric solutions to systems of conservation laws we mention [13,16,19], the latter reference especially for the behavior at the origin.

In this paper, we address system (1.1) in n space dimensions. The new contributions consist in (a) extending the notion of shadow waves to the n-dimensional rotationally symmetric case; (b) constructing nonclassical radially symmetric solutions to the n-dimensional pressureless Euler system, in particular, solutions to the pseudo-Riemann problem, and (c) determining physically meaningful boundary conditions at the origin. (The term pseudo-Riemann problem refers to initial data which are piecewise multiples of $|\vec{x}|^{1-n}$ with a jump at $|\vec{x}| = R > 0$; these arise due to the fact that the radially transformed system contains source terms.)

The plan of exposition is as follows: Section 2 addresses the radially symmetric version of system (1.1) and the appropriate boundary conditions at the origin. In Section 3, we recall the notion of shadow waves, extend it to the radially symmetric, higher dimensional case, and present the basic calculation of the weak limits fixing the structure of the shadow wave. This concerns both the local case (the behavior near the initial jump at $|\vec{x}| = R$) and the global behavior including the origin $\vec{x} = 0$. Section 4 is a brief discussion of the case of solutions with constant wave speed (which will turn out to be the physically meaningful entropy solutions to the pseudo-Riemann problem). Section 5 is devoted to deriving and discussing entropy conditions for shadow wave solutions. Finally, in Section 6 we solve the pseudo-Riemann problem explicitly, obtaining solutions which are physical (i.e., with nonnegative density) and dissipative (i.e., satisfying the entropy condition). Further, these solutions are shown to be locally unique (for small time). In the last subsection we exhibit an example of nonuniqueness, provided by a global, physical solution which violates the entropy condition (and has non-constant wave speed).

2. The radially symmetric pressureless Euler system

As physical background, we consider the system of pressureless gas dynamics (1.1) as describing the evolution of the density $\rho(\vec{x}, t)$ and velocity field $\vec{u}(\vec{x}, t)$ of a gas of sticky particles. Here $t \geq 0$ and $\vec{x} \in \mathbb{R}^n$; ρ is a scalar quantity and $\vec{u}(\vec{x}, t) \in \mathbb{R}^n$. Initial conditions $\rho(\vec{x}, 0) = \rho_0(\vec{x})$ and $\vec{u}(\vec{x}, 0) = \vec{u}_0(\vec{x})$ are presumed to be given.

Assuming that the initial data are radially symmetric it makes sense to search for radially symmetric solutions—indeed it is obvious that classical solutions are radially symmetric if the initial data are. Making