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Improved Moser–Trudinger inequality for functions with mean value zero in $\mathbb{R}^n$ and its extremal functions

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ABSTRACT

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $W^{1,n}(\Omega)$ be the Sobolev space on Ω, and $\lambda(\Omega) = \inf\{\|\nabla u\|_n^n : \int_\Omega u dx = 0, \|u\|_n = 1\}$ be the first nonzero Neumann eigenvalue of the $n-$Laplace operator $-\Delta_n$ on Ω. For $0 \le \alpha < \lambda(\Omega)$, let us define $\|u\|_{1,\alpha}^n = \|\nabla u\|_n^n - \alpha \|u\|_n^n$. We prove, in this paper, the following improved Moser–Trudinger inequality on functions with mean value zero on Ω,

$$\sup_{u\in W^{1,n}(\Omega), \int_\Omega u dx = 0, \|u\|_{1,\alpha}=1} \int_\Omega e^{\beta_n |u|^{\frac{n}{n-1}}} dx < \infty,$$

where $\beta_n = n(\omega_{n-1}/2)^{1/(n-1)}$, and ω_{n-1} denotes the surface area of unit sphere in $\mathbb{R}^n$. We also show that this supremum is attained by some function $u^* \in W^{1,n}(\Omega)$ such that $\int_\Omega u^* dx = 0$ and $\|u^*\|_{1,\alpha} = 1$. This generalizes a result of Ngo and Nguyen (0000) in dimension two and a result of Yang (2007) for $\alpha = 0$, and improves a result of Cianchi (2005).

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $W_0^{1,n}(\Omega)$ be completion of $C_0^\infty(\Omega)$ under the Dirichlet norm $\|u\|_{W_0^{1,n}(\Omega)} = \left(\int_\Omega |\nabla u|^n dx\right)^{1/n}$. The Moser–Trudinger inequality asserts that

$$\sup_{u\in W_0^{1,n}(\Omega), \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha u^{\frac{n}{n-1}}} dx < \infty, \tag{1.1}$$

for any $\alpha \le \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}$ where ω_{n-1} denotes the area of unit sphere in $\mathbb{R}^n$. This inequality (1.1) was proved independently by Poho žaev [21], Yudovič [33] and Trudinger [26]. The sharp constant α_n was found by Moser [19].

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Let $W^{1,n}(\Omega)$ be the completion of $C^\infty(\overline{\Omega})$ under the norm

$$\|u\|_{W^{1,n}(\Omega)} = (\|u\|_n^n + \|\nabla u\|_n^n)^{1/n}.$$

In [6], Cianchi proved a sharp Moser–Trudinger inequality for functions in $W^{1,n}(\Omega)$ with mean value zero as follows

$$\sup_{u\in W^{1,n}(\Omega),\int_\Omega u dx=0,\|\nabla u\|_n\le 1} \int_\Omega e^{\beta|u|^{\frac{n}{n-1}}} dx < \infty, \tag{1.2}$$

for any $\beta \le \beta_n = n(\omega_{n-1}/2)^{1/(n-1)}$. Moreover, if $\beta > \beta_n$ then the supremum in (1.2) will be infinite. In special case when Ω is ball B^n in $\mathbb{R}^n$, the inequality (1.2) was proved by Leckband in [13]. This inequality generalizes an earlier result of Chang and Yang [5] in dimension two,

$$\sup_{u\in W^{1,2}(\Omega),\int_\Omega u dx=0,\|\nabla u\|_2\le 1} \int_\Omega e^{\beta|u|^2} dx < \infty \tag{1.3}$$

for any $\beta \le 2\pi$. A sharpened version of (1.3) in spirit of Adimurthi and Druet [1] was proved by Lu and Yang in [18].

In [20], Ngo and the author proved another sharpened version of Moser–Trudinger type inequality for functions with mean value zero in dimension two. To state the result in that paper, let us denote by

$$\lambda(\Omega) = \inf\left\{\|\nabla u\|_2^2 \,:\, u \in W^{1,2}(\Omega), \|u\|_2 = 1, \int_\Omega u dx = 0\right\}$$

the first nonzero Neumann eigenvalue of $-\Delta$ on Ω, and for $0 \le \alpha < \lambda(\Omega)$, we denote

$$\|u\|_{1,\alpha}^2 = \|\nabla u\|_2^2 - \alpha\|u\|_2^2.$$

In [20], Ngo and the author proved the following inequality,

$$\sup_{u\in W^{1,2}(\Omega),\|u\|_{1,\alpha}\le 1,\int_\Omega u dx=0} \int_\Omega e^{2\pi u^2} dx < \infty. \tag{1.4}$$

This is an improvement of (1.3) in spirit of Tintarev [24] for the classical Moser–Trudinger inequality. Such a result recently was proved for the singular Moser–Trudinger inequality in dimension two by Yang and Zhu [32]. As shown in [20], (1.4) is stronger than the one of Lu and Yang [18] and the one of Chang and Yang (1.3). It is also proved in [20] that the supremum in (1.4) is attained by some functions $u \in W^{1,2}(\Omega)$ with $\int_\Omega u dx = 0$ and $\|u\|_{1,\alpha} \le 1$.

Our goal of this paper is to establish an improvement of type (1.4) for inequality (1.2). Let Ω be a smooth bounded domain in $\mathbb{R}^n$, we denote

$$\mathcal{H} = \left\{u \in W^{1,n}(\Omega) \,:\, \int_\Omega u dx = 0\right\}$$

the subspace of $W^{1,n}(\Omega)$ consisting the functions of mean value zero. Denote

$$\lambda_1(\Omega) = \inf\{\|\nabla u\|_n^n \,:\, u \in \mathcal{H}, \|u\|_n = 1\}$$

the first nonzero Neumann eigenvalue of $n-$Laplace $-\Delta_n$ on Ω. By a simple variational argument, we can prove that $\lambda_1(\Omega)$ is strict positive and is attained by a function in $\mathcal{H}$. For $0 \le \alpha < \lambda_1(\Omega)$, we define

$$\|u\|_{1,\alpha}^n = \|\nabla u\|_n^n - \alpha\|u\|_n^n, \quad u \in \mathcal{H}.$$

Note that $\|\cdot\|_{n,\alpha}$ is a norm on $\mathcal{H}$. Our first main result reads as follows