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The existence of weak solutions for steady flows of electrorheological fluids with nonhomogeneous Dirichlet boundary condition

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ABSTRACT

In this paper, we show the existence of weak solutions for steady flows of electrorheological fluids with nonhomogeneous Dirichlet boundary condition under the condition $p(x) > \frac{2n}{n+2}$, $n = 2, 3$. In particular, we improve *a priori* estimate for weak solutions to the problem with $\min p(x) < 2$, which is an improvement even for constant p.

1. Introduction and the main results

In this paper we investigate the existence of solutions to the system

$$\begin{cases} -\mathrm{div} S(x, \mathcal{D}u) + (u \cdot \nabla)u + \nabla \pi = f, & \text{in } \Omega, \\ \mathrm{div}\, u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u is the velocity, π the pressure, f, g are prescribed functions and $\mathcal{D}u$ is the symmetric part of ∇u. We assume that $S(x,\zeta)$ is a Caratheodory function satisfying the following hypotheses:

$$\exists c_1 > 0, \qquad S(x,\zeta) : \zeta \geq c_1(|\zeta|^{p(x)} - 1), \tag{1.2}$$

$$\exists c_2 > 0, \qquad |S(x,\zeta)| \leq c_2(1 + |\zeta|)^{p(x)-1}, \tag{1.3}$$

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$$(S(x,\zeta) - S(x,\xi)) : (\zeta - \xi) > 0, \quad \forall \zeta, \xi \in S_n(\zeta \neq \xi), \tag{1.4}$$

where $p(x) > 1$ is prescribed function and S_n denotes the set of all symmetric $n \times n$ matrices.

This kind of systems models incompressible electrorheological fluids (ERF) with shear dependent viscosity, which are viscous fluids characterized by their ability to highly change in their mechanical properties when an electric field is applied [21,23]. In fact, the model introduced in [23] have had a big impact into the rapid development of the theory of variable exponent function spaces and of PDEs with variable exponents [1,9,20,24,25].

There are several papers concerning the existence of weak solutions to the problem (1.1) with $g = 0$. The existence of weak solutions to the problem was shown by [23] for $p(x) > \frac{3n}{n+2}$ (n is the spacial dimension) by means of monotone operator theory; this result was improved in [12] for $p(x) > \frac{2n}{n+2}$ by Lipschitz truncation method. For the existence of weak solutions to the problem (1.1) with constant p, we refer to [16]. The authors in [13] showed the same result as in [12] in non-regular domains, by combining a localization of the Lipschitz truncation method with a domain decomposition theorem. In [7], it is shown the existence of weak solutions to a coupled system of the generalized Navier–Stokes equations and convection–diffusion equation with non-linear diffusivity by using a generalization of the monotone operator theory which fits into the framework of generalized Sobolev spaces with variable exponent. Using the methods of Lipschitz truncation and L^∞-truncation, the authors in [15] proved the existence of weak solutions for steady version of a generalized model of ERF, which contains a more realistic description of the dependence of the ER effect on the direction of the electric field. In [6], the authors study construction of solenoidal Lipschitz functions, which enables them to prove the existence of weak solutions to the two-dimensional Prandtl–Eyring fluid model for which a correction via Bogovskii operators does not work. For regularity of solutions to the problem (1.1), we refer [3,5,8,10,14,26].

Throughout this paper, let Ω be a bounded locally Lipschitz domain of $\mathbb{R}^n$, $n = 2, 3$. The natural necessary condition imposed on the boundary data g by the incompressibility property only requires that

$$\int_{\partial\Omega} g \cdot \nu ds = 0, \tag{1.5}$$

where ν is the unit outer normal on $\partial\Omega$. Now, it is an outstanding open question to prove or disprove existence of solutions to the Navier–Stokes equations with nonhomogeneous Dirichlet boundary conditions when only the natural compatibility condition (1.5) is satisfied. The existence of weak solution to the Navier–Stokes equations is shown provided the data g (belongs to a suitable function class and) satisfies the conditions

$$\int_{\Gamma_i} g \cdot \nu \, ds = 0, \quad i = 1, \ldots, k, \tag{1.6}$$

where Γ_i are the k connected components of the boundary $\partial\Omega$ [17,19].

Galdi [18] proved that the problem (1.1) with constant $p > 2$ is solvable under only the assumption (1.5). We note that the situation is even worse if $p < 2$, see the argument following Theorem 1.3. For the problem (1.1) with $p = const < 2$, if g satisfying (1.5) is small in a suitable norm, then a weak solution exists for $p > \frac{3n}{n+2}$ and very weak solution for $p > \frac{2n}{n+2}$ [4].

To our knowledge, there are no results about the existence of weak solutions for steady flows of electrorheological fluids with nonhomogeneous Dirichlet boundary conditions.

In this paper, we prove the existence of weak solutions to the problem (1.1) under the restriction $p(x) > \frac{2n}{n+2}$, $n = 2, 3$ under various conditions on boundary data g according to the ranges of $p_- = \min p(x)$; we give various conditions on g according to the ranges of p_- in order to absorb the convective term to leading term.

Throughout the paper, we assume the followings:

$$\Omega \in C^{0,1}; \qquad p \in \mathscr{P}^{log}(\overline{\Omega}); \qquad f \in W^{-1,p'(x)}(\Omega); \qquad g \in TrW^{1,p(x)}(\Omega), \tag{1.7}$$