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On the lower semicontinuous envelope of functionals defined on polyhedral chains

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ABSTRACT

In this note we prove an explicit formula for the lower semicontinuous envelope of some functionals defined on real polyhedral chains. More precisely, denoting by $H : \mathbb{R} \to [0,\infty)$ an even, subadditive, and lower semicontinuous function with $H(0) = 0$, and by Φ_H the functional induced by H on polyhedral m-chains, namely

$$\Phi_H(P) := \sum_{i=1}^{N} H(\theta_i)\mathcal{H}^m(\sigma_i), \quad \text{for every } P = \sum_{i=1}^{N} \theta_i [\![\sigma_i]\!] \in \mathbf{P}_m(\mathbb{R}^n),$$

we prove that the lower semicontinuous envelope of Φ_H coincides on rectifiable m-currents with the H-mass

$$\mathbb{M}_H(R) := \int_E H(\theta(x))\, d\mathcal{H}^m(x) \quad \text{for every } R = [\![E,\tau,\theta]\!] \in \mathbf{R}_m(\mathbb{R}^n).$$

1. Introduction

Let $H : \mathbb{R} \to [0,\infty)$ be an even, subadditive, and lower semicontinuous function, with $H(0) = 0$. The function H naturally induces a functional Φ_H on the set of polyhedral m-chains in $\mathbb{R}^n$, which can be thought as the space of linear combinations of m-simplexes with real coefficients. For every polyhedral m-chain of the form $P = \sum_{i=1}^{N} \theta_i [\![\sigma_i]\!]$ (with non-overlapping m-simplexes σ_i), we set

$$\Phi_H(P) := \sum_{i=1}^{N} H(\theta_i)\mathcal{H}^m(\sigma_i).$$

It is easy to see that the above assumptions on H are necessary for the functional Φ_H to be (well defined and) lower semicontinuous on polyhedral chains with respect to convergence in flat norm. In this note, we

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prove that they are also sufficient, and moreover we show that the lower semicontinuous envelope of Φ_H coincides on rectifiable m-currents with the H-mass, namely the functional

$$\mathbb{M}_H(R) := \int_E H(\theta(x))\, d\mathcal{H}^m(x), \qquad \text{for every rectifiable } m\text{-current } R = [\![E,\tau,\theta]\!].$$

The validity of such a representation has recently attracted some attention. For instance, it is clearly assumed in [17] for the choice $H(x) = |x|^\alpha$, with $\alpha \in (0,1)$, in order to prove some regularity properties of minimizers of problems related to branched transportation (see also [3,12,13]) and in [5] in order to define suitable approximations of the Steiner problem, with the choice $H(x) = (1+\beta|x|)\mathbf{1}_{\mathbb{R}\setminus\{0\}}$, where $\beta > 0$ and $\mathbf{1}_A$ denotes the indicator function of the Borel set A. Our results were recently used in [4] to define and study branched minimizers with respect to the most general class of reasonable transportation costs.

We finally remark that in the last section of [15] the author sketches a strategy to prove an analogous version of the main theorem of the paper (Theorem 2.4) in the framework of flat chains with coefficients in a normed abelian group G. Motivated by the relevance of such result for real valued flat chains, the ultimate aim of our note is to present a self-contained complete proof of it when $G = \mathbb{R}$.

2. Notation and main result

If $0 \le m \le n$, then compactly supported m-dimensional currents, rectifiable m-currents, polyhedral m-chains, and flat m-chains in $\mathbb{R}^n$ with real coefficients will be denoted $\mathscr{E}_m(\mathbb{R}^n)$, $\mathbf{R}_m(\mathbb{R}^n)$, $\mathbf{P}_m(\mathbb{R}^n)$ and $\mathbf{F}_m(\mathbb{R}^n)$, respectively. In what follows, we briefly recall the relevant definitions of the above classes of currents; for the basic definitions about currents, such as the boundary operator ∂, the support spt, and the mass norm $\mathbb{M}$, we refer the reader to [14]. Let us denote by $\Lambda^m(\mathbb{R}^n)$ the vector space of m-covectors in $\mathbb{R}^n$. A current R is in $\mathbf{R}_m(\mathbb{R}^n)$ if its action on any differential m-form $\omega \in \mathscr{D}^m(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n; \Lambda^m(\mathbb{R}^n))$ can be expressed by

$$\langle R, \omega\rangle = \int_E \langle \omega(x), \tau(x)\rangle\, \theta(x)\, d\mathcal{H}^m(x), \tag{2.1}$$

where $E \Subset \mathbb{R}^n$ is countably m-rectifiable, $\tau(x)$ is an $\mathcal{H}^m$-measurable, unit, simple m-vector field orienting the approximate tangent space $\mathrm{Tan}(E,x)$ at $\mathcal{H}^m$-a.e. $x \in E$, and $\theta \in L^1(\mathcal{H}^m \llcorner E; (0,\infty))$ is a positive-valued multiplicity. If R is given by (2.1), we will write $R = [\![E,\tau,\theta]\!]$. We remark that the rectifiable currents we are considering all have finite mass and compact support. A polyhedral chain $P \in \mathbf{P}_m(\mathbb{R}^n)$ is a rectifiable current which can be written as a linear combination

$$P = \sum_{i=1}^N \theta_i [\![\sigma_i]\!], \tag{2.2}$$

where $\theta_i \in (0,\infty)$, the σ_i's are non-overlapping, oriented, m-dimensional, convex polytopes (finite unions of m-simplexes) in $\mathbb{R}^n$ and $[\![\sigma_i]\!] = [\![\sigma_i,\tau_i,1]\!]$, τ_i being a constant m-vector orienting σ_i. If $P \in \mathbf{P}_m(\mathbb{R}^n)$, then its *flat norm* is defined by

$$\mathbb{F}(P) := \inf\{\mathbb{M}(S) + \mathbb{M}(P - \partial S) \,:\, S \in \mathbf{P}_{m+1}(\mathbb{R}^n)\}.$$

Flat m-chains can be therefore defined to be the $\mathbb{F}$-*completion* of $\mathbf{P}_m(\mathbb{R}^n)$ in $\mathscr{E}_m(\mathbb{R}^n)$.

We remark that for the spaces of currents considered above the following chain of inclusions holds:

$$\mathbf{P}_m(\mathbb{R}^n) \subset \mathbf{R}_m(\mathbb{R}^n) \subset \mathbf{F}_m(\mathbb{R}^n) \cap \{T \in \mathscr{E}_m(\mathbb{R}^n) \,:\, \mathbb{M}(T) < \infty\}. \tag{2.3}$$

The flat norm $\mathbb{F}$ extends to a functional (still denoted $\mathbb{F}$) on $\mathscr{E}_m(\mathbb{R}^n)$, which coincides on $\mathbf{F}_m(\mathbb{R}^n)$ with the completion of the flat norm on $\mathbf{P}_m(\mathbb{R}^n)$, by setting:

$$\mathbb{F}(T) := \inf\{\mathbb{M}(S) + \mathbb{M}(T - \partial S) \,:\, S \in \mathscr{E}_{m+1}(\mathbb{R}^n)\}. \tag{2.4}$$