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Fractional Schrödinger–Poisson–Kirchhoff type systems involving critical nonlinearities

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ABSTRACT

The paper is concerned with existence, multiplicity and asymptotic behavior of nonnegative solutions for a fractional Schrödinger–Poisson–Kirchhoff type system. As a consequence, the results can be applied to the special case

$$\left(a+b\|u\|^2\right)\left[(-\Delta)^s u+V(x)u\right]+\phi k(x)|u|^{p-2}u=\lambda h(x)|u|^{q-2}u+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3,$$
$$(-\Delta)^t\phi=k(x)|u|^p \quad \text{in } \mathbb{R}^3,$$
$$\|u\|=\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}V(x)|u|^2dx\right)^{1/2},$$

where $a,b\geq 0$ are two numbers, with $a+b>0$, $1<p<2_{s,t}^*=\frac{3+2t}{3-2s}$, $\lambda>0$ is a parameter, $s,t\in(0,1)$, $(-\Delta)^s$ is the fractional Laplacian, $k\in L^{\frac{6}{3+2t-p(3-2s)}}(\mathbb{R}^3)$ may change sign, $V:\mathbb{R}^3\to[0,\infty)$ is a potential function, $2_s^*=6/(3-2s)$ is the critical Sobolev exponent, $1<q<2_s^*$ and $h\in L^{\frac{2_s^*}{2_s^*-q}}(\mathbb{R}^3)$. First, when $\theta<p<2_s^*/2$, $2p\leq q<2_s^*$ and λ is large enough, existence of nonnegative solutions is obtained by the mountain pass theorem. Moreover, we obtain that $\lim_{\lambda\to\infty}\|u_\lambda\|=0$. Then, via the Ekeland variational principle, existence of nonnegative solutions is investigated when $\theta<p<2_s^*/2$, $1<q<2$ and λ is small enough, and we obtain that $\lim_{\lambda\to 0}\|u_\lambda\|=0$. Finally, we consider the system with double critical exponents, that is, $p=2_{s,t}^*$, and obtain two nontrivial and nonnegative solutions in which one is least energy solution and another is mountain pass solution. The paper covers a novel feature of Kirchhoff problems, which is the fact that the parameter a can be zero. Hence the results of the paper are new even for the standard stationary Schrödinger–Poisson–Kirchhoff system.

1. Introduction and main results

In this paper, we study the following fractional Schrödinger–Poisson–Kirchhoff type system:

$$\begin{cases} M(\|u\|^2)[(-\Delta)^s u+V(x)u]+\phi k(x)|u|^{p-2}u=\lambda h(x)|u|^{q-2}u+|u|^{2_s^*-2}u & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=k(x)|u|^p \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where

$$\|u\| = \left([u]_s^2 + \int_{\mathbb{R}^3} V(x)|u|^2 dx\right)^{1/2}, \quad [u]_s = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy\right)^{1/2}, \tag{1.2}$$

$s,t \in (0,1)$, $M : [0,\infty) \to [0,\infty)$ is a Kirchhoff function, $1 < p < \frac{3+2t}{3-2s}$, $1 < q < 2_s^*$, $\lambda > 0$, $h \in L^{\frac{2_s^*}{2_s^*-q}}(\mathbb{R}^3)$, $k \in L^{\nu}(\mathbb{R}^3)$, $\nu = \frac{6}{3+2t-p(3-2s)}$, $V : \mathbb{R}^3 \to [0,\infty)$ is a continuous function, $2_s^* = 6/(3-2s)$ is the critical fractional Sobolev exponent and $(-\Delta)^s$ is the fractional Laplace operator which, up to a normalization constant, is defined as

$$(-\Delta)^s \varphi(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^3)$. Henceforward $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ and radius $\varepsilon > 0$. For details on fractional Laplace operator we refer the readers to [21] and the references therein.

A typical example of M is given by $M(t) = a + b\,t^{\theta-1}$ for $t \geq 0$, where $a, b \geq 0$ and $a + b > 0$, if $\theta > 1$, and $M(t) = a > 0$ if $\theta = 1$. For $\theta > 1$, when M is of this type, problem (1.1) is said to be *non-degenerate* if $a > 0$, while it is called *degenerate* if $a = 0$.

When $M \equiv 1$ and $k \equiv 0$, system (1.1) reduces to the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda h(x)|u|^{q-2}u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

which is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes [29,30]. See also [20] for a detailed mathematical description of fractional Schrödinger equation. In recent years, nonlocal operators and related equations have been receiving a great attention. Actually, nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [3]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media, for more details see [3,13,14] and the references therein. Indeed, the literature on nonlocal fractional operators and on their applications is quite large, see for example the recent monograph [43], the extensive paper [22] and the references cited therein.

When $s, t \to 1, M \equiv 1, K \equiv 1$ and $p = 2$, system (1.1) becomes the Schrödinger–Poisson type system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

which has been first introduced by Benci and Fortunato [8] as a physical model describing solitary waves for nonlinear Schrödinger type equations interacting with an unknown electrostatic field. The first equation of (1.4) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field. For more details about the physical background of the system (1.4), we refer the readers to [9,35,40] and the references cited there.

In the last decades, many papers have been devoted to the existence and multiplicity of solutions for system like (1.4) via variational methods and critical point theory under various assumptions on the potential V and the nonlinearity, see for example [26,27,31,38,51]. In [36], Liu considered the following generalized Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u - k(x)\phi|u|^3 u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = k(x)|u|^5 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$