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Hardy space estimates of Hausdorff operators on the Heisenberg group

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ABSTRACT

This is a continuation of a series of works about Hausdorff operator on Hardy spaces in the setting of the Heisenberg group. In this paper, we obtain the boundedness from Lebesgue spaces to Hardy spaces for fractional Hausdorff operators and their compositions with Riesz transforms. We also establish the boundedness for two kinds of special Hausdorff operators, the Hausdorff–Poisson operator and the Hausdorff–Gauss operator, on Hardy spaces.

1. Introduction

The theory of Hausdorff operator originates from the classical analysis and it is closely connected to the study of the Fourier analysis. The 1-dimensional Hausdorff operator (Hausdorff summability methods) appeared long time ago aiming to solve certain classical problems in analysis. Modern theory of Hausdorff operator started with the work of Siskakis in the setting of complex analysis and with the work of Liflyand–Móricz in the setting of Fourier transform. For more information about the background and the historical development of the subject, the reader can refer to [24,26–28] and the references therein.

Fix an appropriate function Φ on $\mathbb{R}^+$, the 1-dimensional Hausdorff operator [4,28] is defined formally by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \quad x \in \mathbb{R}.$$

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The definition looks quite simple with employing the dilation structure of $\mathbb{R}$, but many classical operators in analysis are special cases of h_Φ if one chooses suitable functions Φ in the above integral (see [4,6,7,10,29]). These operators include Hardy operators, Cesáro operators, Hilbert operators, Hardy–Littlewood–Pólya operators and many others [8,13,20,31]. Also, the Riemann–Liouville fractional integral and the Weyl fractional integral can be derived from the Hausdorff operator. On the other hand, we notice that the so called weighted Hardy operator $\mathcal{H}_\psi(f)$ is another popular operator that recently attracts some research works (see [2,12,14,15,19,22,32,37,39], etc.). However, the operator $\mathcal{H}_\psi(f)$ actually is also a special kind of Hausdorff operator. To see this fact, we note that the definition of $h_\Phi(f)(x)$ clearly makes sense not only for $x > 0$, but also for $x \in \mathbb{R}^n$. In this latter setting if we set $\Phi(t) = t^{-1}\psi(t^{-1})\chi_{(1,\infty)}(t)$ and change variables $t^{-1} = u$, then $h_\Phi(f)(x)$ is reduced to the weighted Hardy operator

$$\mathcal{H}_\psi(f)(x) = \int_0^1 \psi(u)f(ux)du, \quad x \in \mathbb{R}^n.$$

The active research on the one dimensional theory naturally inspires people to study the Hausdorff operator in high dimensional spaces. In Euclidean space $\mathbb{R}^n$, $n \geq 2$, one of the analogs of h_Φ is the operator

$$H_\Phi(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy.$$

Generally, we can define the fractional Hausdorff operator

$$H_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{n-\beta}} f\left(\frac{x}{|y|}\right) dy, \ \beta \geq 0.$$

Here, as well as in the following content, we always assume that f is initially a function in the Schwartz class S for the sake of simplicity. We notice that, like the one dimension operator h_Φ, both operators H_Φ and $H_{\Phi,\beta}$ have received extensive study in recent years [1,4–7,20,24,25,27,28,33]. Particularly, as an interesting upgrade of the Hausdorff operator of Euclidean space version , we begin to study the Hausdorff operator in a more general underlying space $\mathbb{H}^n$, the Heisenberg group [34]. It is known that the Heisenberg group is not only a special homogeneous group [11], but it also can be viewed as the Shilov boundary of the Siegel domain in several complex analysis. Thus, the Heisenberg group plays significant roles in many branches of mathematics, such as representation theory, harmonic analysis, several complex variables, partial differential equations, and quantum mechanics; see [17] and [36] for more details. Additionally, the Heisenberg group is widely applied in signal theory and many related topics (see [35]).

As we discussed in [34], the Heisenberg group $\mathbb{H}^n$ is a non-commutative nilpotent Lie group and the geometric motions on $\mathbb{H}^n$ are quite different from those on $\mathbb{R}^n$ due to the loss of interchangeability. But $\mathbb{H}^n$ inherits the dilation structure of the Euclidean space. This nice structure is a perfect fusion with the definition of the Hausdorff operator. Additionally, $\mathbb{H}^n$ is a space of homogeneous type in the sense of Coifman–Weiss, so that it possesses both metric and measure and satisfies the doubling property. These features are enough for us to study the boundedness of Hausdorff operator on Banach function spaces with $\mathbb{H}^n$ as underlying space, see [34]. The main purpose of this article is to further explore, on the Heisenberg group $\mathbb{H}^n$, the boundedness of the Hausdorff operator on the Hardy space H^p. It is known that the H^p space is merely a quasi-normed space and the study of Hausdorff operator on H^p for $0 < p < 1$ is one of the most difficult parts in the theory of Hausdorff operator. Firstly, Liflyand and Miyachi in [27] observed that in order to obtain the boundedness of $h_\Phi(f)$ on H^p for $0 < p < 1$, the assumption of the smoothness of Φ is necessarily needed, so that in this case one must look for a totally different method from that used in the study of boundedness of h_Φ on Banach spaces such as on L^p or H^1 (see [20,27,29]). Secondly we find that all methods in [20,27,29] to treat the boundedness on $H^p(\mathbb{R})$ for $0 < p < 1$ fail when we proceed to the $H^p(\mathbb{R}^n)$ boundedness of H_Φ when $0 < p < 1$ and $n \geq 2$. Observing that h_Φ can be equivalently defined as

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(x/t)}{t} f(t)\, dt, \quad x \geq 0,$$