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Boundary stabilization for wave equations with damping only on the nonlinear Wentzell boundary[☆]

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A B S T R A C T

We are concerned with the problem of boundary stabilization for wave equations with damping only on the nonlinear Wentzell boundary, and the damping is also nonlinear. We obtain boundary stabilization results (explicit energy decay rates), as well as the wellposedness, for the systems. Our method consists in establishing a new type of non-uniform integral inequality of energy, and exploiting it to derive the decay rates. The results are set up in all space dimensions. Even for the one dimensional linear system, the result is new, and we also indicate that the obtained decay rate is optimal in this case, that is, the obtained decay rate for linear strings is optimal.

1. Introduction

In this paper, we investigate the following problem of boundary stabilization for wave equations with damping only on the nonlinear dynamical boundary:

$$u_{tt} - a\Delta u = 0, \qquad \text{in } \Omega,\ t \geq 0, \tag{1.1}$$

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$$u = 0, \qquad \text{on } \Gamma_0,\ t \geq 0, \tag{1.2}$$
$$\rho_0 u_{tt} + \sigma \partial_\nu u + g(u_t) + f(u) = 0, \qquad \text{on } \Gamma_1,\ t \geq 0, \tag{1.3}$$
$$u(0,x) = u_0,\ u_t(0,x) = u_1, \qquad \text{in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, which is divided into two disjoint parts

$$\Gamma = \Gamma_0 \cup \Gamma_1,$$

each being closed and nonempty, and $\nu(x)$ is the unit outer normal vector at boundary. The equality (1.3) is a nonlinear dynamical boundary condition. For the case of a string ($n = 1$) or membrane ($n = 2$),

- u represents the displacement,
- a is the wave speed,
- ρ_0 stands for the boundary density,
- σ is the tension force per unit length,
- $\sigma \partial_\nu u$ represents the tension force from the interior to the boundary,
- $g(u_t)$ is the nonlinear velocity feedback control,
- $f(u)$ stands for the nonlinear force,

and (1.1)–(1.3) models the vertical motion of stretched strings with tip masses attached to the endpoints or membranes with thin boundary layers of high rigidity.

The equality (1.3) is also a nonlinear Wentzell boundary condition (simplified Wentzell BC, not containing the term of tangential Laplacian compared with [6,7,16]). Without loss of generality, we assume $a = 1, \rho_0 = 1, \sigma = 1$ from now on.

In the absence of the boundary term u_{tt} on Γ_1, (1.3) is a nonlinear Neumann boundary condition:

$$\partial_\nu u + g(u_t) + f(u) = 0, \quad \text{on } \Gamma_1,\ t \geq 0. \tag{1.5}$$

One knows from [18] that the solution energies of the system (1.1) with (1.2) and (1.5) decay uniformly under suitable conditions on g and f, in particular, decay exponentially when the damping $g(u_t)$ is linear. In the presence of the term u_{tt} on Γ_1, things become quite different. Even for the case of a linear string ($n = 1$, $f = 0$, and $g(s) = s$), the energies for system (1.1)–(1.3) are no longer stable in a uniform way, although each approaches zero (see [21]); recently, it is shown (see [22]) for this case that there exists a weak solution whose energy decays at an arbitrarily prescribed slow rate.

In [28], the authors added an angular velocity term at the dynamical boundary point $\Gamma_1 = \{1\}$, so that the combined feedback control force, like $u_{xt}(t,1) + u_t(t,1)$, is able to stabilize the system uniformly. Also, the uniform stability was established by means of both boundary and internal dampings in [6,7,16], where $n \geq 2$ and a tangential Laplacian is also contained in the dynamical boundary condition modeling the tension of the boundary itself or the heat source located on the boundary and interacting with a heat flux in the surrounding area (cf. [14]). We refer the reader to [3,8–13,15,24–27] and the references therein for the related works. Moreover, for the relevant approaches used in the study of acoustic boundary problems, which is also a class of dynamical boundary value problems, we refer the reader to, e.g., [1,2].

The aim of the present paper is to obtain boundary stabilization results, i.e., explicit energy decay rates for the classical (strong) solutions of system (1.1)–(1.3) under suitable conditions on g and f, without the help of either boundary angular velocity damping or internal damping. This turns out to be difficult (see also [6, Remark A.1]), owing to the problem of how to bring the high-order terms on the boundary under control.

The paper is structured by four sections. In Section 2, we show the wellposedness for the initial–boundary value problem (1.1)–(1.4), by the use of the nonlinear semigroup theory. Section 3 is devoted to deriving decay rates of the energies for the system. Our approach is mainly divided into two steps: