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Bifurcation of positive solutions for a one-dimensional indefinite quasilinear Neumann problem

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ABSTRACT

We study the structure of the set of the positive regular solutions of the one-dimensional quasilinear Neumann problem involving the curvature operator

$$-\left(u'/\sqrt{1+(u')^2}\right)' = \lambda a(x) f(u), \quad u'(0)=0,\ u'(1)=0.$$

Here $\lambda \in \mathbb{R}$ is a parameter, $a \in L^1(0,1)$ changes sign, and $f \in \mathcal{C}(\mathbb{R})$. We focus on the case where the slope of f at 0, $f'(0)$, is finite and non-zero, and the potential of f is superlinear at infinity, but also the two limiting cases where $f'(0) = 0$, or $f'(0) = +\infty$, are discussed. We investigate, in some special configurations, the possible development of singularities and the corresponding appearance in this problem of bounded variation solutions.

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1. Introduction

The main goal of this paper is analyzing the positive regular solutions of the quasilinear Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x) f(u), \quad 0<x<1, \\ u'(0)=u'(1)=0, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^1(0,1)$, and $f \in \mathcal{C}(\mathbb{R})$. By a regular solution we mean a function $u \in W^{2,1}(0,1)$ which satisfies the equation a.e. in $(0,1)$ and the Neumann conditions $u'(0)=u'(1)=0$. We also assume that the weight a changes sign, f vanishes at 0 and is strictly increasing, and the potential of f,

$$F(u) := \int_0^u f(s)\, ds, \tag{1.2}$$

is superlinear at infinity. As we shall see, under these assumptions on f, the existence of a positive solution of (1.1) entails that the sign of a must change. This research is also motivated by the large amount of studies devoted to the existence of positive solutions for semilinear elliptic problems with indefinite nonlinearities, that started nearly three decades ago with [6,1,2,8,7,3] and since then have had a tremendous development in several different directions.

This problem is a special one-dimensional counterpart of the elliptic problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h(x,u), \quad \text{in } \Omega, \\ -\dfrac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which plays a relevant role in the mathematical analysis of various physical or geometrical issues, such as when describing capillarity phenomena for incompressible fluids, or modeling reaction–diffusion processes where the flux response to an increase of gradients slows down and ultimately approaches saturation at large gradients, or studying prescribed mean curvature problems for Cartesian graphs in the Euclidean space; significant references related to these topics include [34,51,9,16,25,21,29,27,24,30,31,28,33,13].

It is a well established fact that introducing the mean curvature operator

$$-\frac{1}{N} \operatorname{div}\left(\nabla u / \sqrt{1+|\nabla u|^2}\right)$$

determines a deep impact on the morphology of the solution patterns of elliptic problems, the most notable of which is the possibility of discontinuous equilibrium states [33,10,11,45,13,47,18,17]. Accordingly, the space of bounded variation solutions is usually considered as an appropriate framework where settling problem (1.3), and hence a suitable notion of solution, involving a variational inequality, has been introduced and systematically used in, e.g., [44,35,45–49,18,41]. It was also noticed in [46] that, by virtue of the results in [5], such definition, when referred to (1.1), can be reformulated as follows: a function $u \in BV(0,1)$ is a bounded variation (BV, for short) solution of (1.1) if

$$\int_0^1 \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx + \int_0^1 \operatorname{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|}\, |D\phi|^{\mathrm{s}} = \int_0^1 a\, f(u)\, \phi\, dx,$$