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Integro-partial differential equations with singular terminal condition

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ABSTRACT

In this paper, we show that the minimal solution of a backward stochastic differential equation gives a probabilistic representation of the minimal viscosity solution of an integro-partial differential equation both with a singular terminal condition. Singularity means that at the final time, the value of the solution can be equal to infinity. Different types of regularity of this viscosity solution are investigated: Sobolev, Hölder or strong regularity.

1. Introduction

The notion of backward stochastic differential equations (BSDEs) was first introduced by Bismut in [10] in the linear setting and by Pardoux & Peng in [32] for nonlinear equation. One particular interest for the study of BSDE is the application to partial differential equations (PDEs). Indeed as proved by Pardoux & Peng in [33], BSDEs can be seen as generalization of the Feynman–Kac formula for nonlinear PDEs. Roughly speaking, if we can solve a system of two SDEs with one forward in time and one backward in time, then the solution is a deterministic function and is a (weak) solution of the related PDE. This is a method of characteristics to solve the parabolic PDE. The converse assertion can be proved provided we can apply Itô's formula, that is if the solution of the PDE is regular enough. Since then a large literature has been developed on this topic (see in particular the books [15,17,34] and the references therein). The extension to quasi-linear PDEs or to fully nonlinear PDEs has been already developed (see among other [29,42]).

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Here we are interesting in another development of the theory: the case of integro-partial differential equation (IPDE) and of (backward) SDEs with Poisson random noise. In [5], Barles, Buckdahn & Pardoux show that we can add in the system of forward backward SDE a Poisson random measure and if we can find a solution to this system, again the solution is a weak solution of an IPDE:

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u,(\nabla u)\sigma,\mathcal{B}(t,x,u)) = 0 \tag{1}$$

with terminal condition $u(T,.) = g$. Here $\mathcal{L}$ is a local second-order differential operator corresponding to the infinitesimal generator of the continuous part of the forward SDE and $\mathcal{I}$ and $\mathcal{B}$ are two integro-differential operators. $\mathcal{I}$ is the discontinuous part of the infinitesimal generator of the forward SDE, and $\mathcal{B}$ is related to the generator of the BSDE. In [5], weak solution means viscosity solution. Since this paper, several authors have weaken the assumptions of [5]. The book [15] (Chapter 4) gives a nice review of these results (and several references on this topic).

Among all semi-linear PDEs, a particular form has been widely studied:

$$\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^q = 0. \tag{2}$$

Baras & Pierre [3], Marcus & Veron [30] (and many other papers) have given existence and uniqueness results for this PDE. In [30] it is shown that every positive solution of (2) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^d$ and μ a non negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}$. The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^m,\ A \text{ Borel},\quad \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view Dynkin & Kuznetsov [16] and Le Gall [28] have proved similar results for the PDE (2) in the case $0 < q \leq 1$ using the theory of superprocesses. Now if we want to represent the solution u of (2) using a FBSDE, we have to deal with a *singular* terminal condition ξ in the BSDE, which means that $\mathbb{P}(\xi = +\infty) > 0$. This singular case and the link between the solution of the BSDE with singular terminal condition and the viscosity solution of the PDE (2) have been studied first in [37]. Recently it was used to solve a stochastic control problem for portfolio liquidation (see [2] or [22]). In [26] we enlarge the known results on this subject for more general generator f (than $f(y) = -y|y|^q$).

In this paper our goal is to generalize the results of [37] and using our recent papers [26,38] we want to study the related IPDE (1) when the terminal condition $u(T,.) = g$ is *singular* in the sense that g takes values in $\mathbb{R}_+ \cup \{+\infty\}$ and the set

$$\mathcal{S} = \{x \in \mathbb{R}^d,\ g(x) = +\infty\}$$

is a non empty closed subset of $\mathbb{R}^d$. Again in the non singular case, if the terminal function g is of linear growth, the relation between the FBSDE and the IPDE is obtained in [5]. Moreover several papers have studied the existence and the uniqueness of the solution of such IPDE (see among others [1,8,9] or [23]).

- To our best knowledges **the study of (1) with a singularity at time** T is completely new. There is no probabilistic representation of such IPDE using superprocesses and no deterministic or analytic works on this topic. In the Ph.D. thesis of Piozin [36] (as in [37]), we have studied the case when $f(t,y,z,u) = f(y) = -y|y|^q$. Hence the aim of the paper is to prove that this minimal solution Y of the singular BSDE is the probabilistic representation of the minimal positive viscosity solution u of the IPDE for general function f with a singular terminal condition.