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Strongly stable surfaces in sub-Riemannian 3-space forms

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ABSTRACT

A surface of constant mean curvature (CMC) equal to H in a sub-Riemannian 3-manifold is strongly stable if it minimizes the functional area $+\,2H$ volume up to second order. In this paper we obtain some criteria ensuring strong stability of surfaces in Sasakian 3-manifolds. We also produce new examples of C^1 complete CMC surfaces with empty singular set in the sub-Riemannian 3-space forms by studying those ones containing a vertical line. As a consequence, we are able to find complete strongly stable non-vertical surfaces with empty singular set in the sub-Riemannian hyperbolic 3-space $\mathbb{M}(-1)$. In relation to the Bernstein problem in $\mathbb{M}(-1)$ we discover strongly stable C^∞ entire minimal graphs in $\mathbb{M}(-1)$ different from vertical planes. These examples are in clear contrast with the situation in the first Heisenberg group, where complete strongly stable surfaces with empty singular set are vertical planes. Finally, we analyze the strong stability of CMC surfaces of class C^2 and non-empty singular set in the sub-Riemannian 3-space forms. When these surfaces have isolated singular points we deduce their strong stability even for variations moving the singular set.

1. Introduction

Let M be a Sasakian sub-Riemannian 3-manifold (to be defined in Section 2.1). From the first variation formulas, see for instance [25, Sect. 4.1], a surface Σ in M with $\partial\Sigma = \emptyset$ which is a critical point of the (sub-Riemannian) area A for any variation preserving the associated volume V has constant mean curvature H in the sense of (3.1). From here, it is easy to deduce that Σ satisfies $(A + 2HV)'(0) = 0$ for any variation. Following standard terminology we will say that Σ is *strongly stable* if, furthermore, we have $(A + 2HV)''(0) \geqslant 0$ for any variation. In particular, it is clear that $A''(0) \geqslant 0$ under volume-preserving variations. In the minimal case ($H = 0$) the strong stability is the classical condition that $A''(0) \geqslant 0$ for any variation.

In recent years constant mean curvature (CMC) surfaces, stability properties and Bernstein type problems have been extensively investigated in sub-Riemannian manifolds. The present paper aims to study these

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topics inside Sasakian 3-manifolds, focusing on the simplest and most symmetric ones: the *space forms*, defined in Section 2.2 as complete Sasakian sub-Riemannian 3-manifolds of constant Webster scalar curvature κ. In the simply connected case, a result of Tanno [34] establishes that the space form $\mathbb{M}(\kappa)$ is, up to isometries, the first Heisenberg group $\mathbb{H}^1$ for $\kappa = 0$, the group of unit quaternions $\mathbb{S}^3 \subset \mathbb{R}^4$ for $\kappa = 1$, and the universal cover of the special linear group $\mathrm{SL}(2,\mathbb{R})$ for $\kappa = -1$. As the authors showed in [25, Sect. 2.2], standard arguments in Riemannian geometry produce 3-dimensional space forms with non-trivial topology.

The analysis of the stability condition requires an explicit expression for $(A + 2HV)''(0)$. The second variation of the sub-Riemannian area has appeared in several contexts, see for instance [9,3,12,26,28,23,8,22, 17,27,20]. The computation of $A''(0)$ by differentiation under the integral sign in (2.4) involves a technical problem since the deformation could move the *singular set* Σ_0, which consists of the points in Σ where the integrand $|N_h|$ vanishes. In [25] we solved this difficulty by assuming the employed deformations to be *admissible*, see Section 3.2 for a precise definition. This allows to apply the Leibniz's rule for differentiating under the integral sign. For a CMC surface Σ inside a Sasakian sub-Riemannian 3-manifold M, we were able to prove that

$$(A + 2HV)''(0) = \mathcal{Q}(w,w) + \int_{\Sigma} \mathrm{div}_{\Sigma}\, G\, da, \tag{1.1}$$

for any admissible variation under suitable integrability conditions. In the previous formula, w denotes the normal component of the velocity vector field associated to the variation, $\mathrm{div}_{\Sigma} G$ is the divergence relative to Σ of a certain tangent vector field G along $\Sigma - \Sigma_0$, see [25, Eq. (7.1)], and $\mathcal{Q}$ is the (sub-Riemannian) *index form* of Σ defined in (3.3). For the particular case of variations supported on $\Sigma - \Sigma_0$ the divergence term vanishes, and we can use integration by parts to deduce

$$(A + 2HV)''(0) = -\int_{\Sigma} w\, \mathcal{L}(w)\, da,$$

where $\mathcal{L}$ is the second order differential operator given in (3.6). This operator plays in our setting the same role as the *Jacobi operator* introduced by Barbosa, do Carmo and Eschenburg [2] for CMC hypersurfaces in Riemannian manifolds. By analogy with the Riemannian situation we define a (sub-Riemannian) *Jacobi function* on Σ as a function $\psi \in C^2(\Sigma)$ for which $\mathcal{L}(\psi) = 0$.

The second variation formula provides a bridge between the stability properties of Σ and the operator $\mathcal{L}$. There is a vast literature exploring this connection in the Riemannian context. A relevant result in this line is a theorem of Fischer-Colbrie and Schoen [16] asserting that a CMC hypersurface having a positive Jacobi function is strongly stable. Recently, Montefalcone [27] has derived a sub-Riemannian counterpart of this theorem for minimal hypersurfaces with empty singular set in Carnot groups. In Section 3 of the present paper we provide a similar stability criterion for a CMC surface Σ with $\Sigma_0 = \emptyset$ in a Sasakian sub-Riemannian 3-manifold. Indeed, from the second variation formula (1.1) and the expression in Eq. (3.11) for the index form $\mathcal{Q}$, we show in Theorem 3.5 strong stability of Σ provided there is a nowhere vanishing function $\psi \in C^2(\Sigma)$ such that $\psi\, \mathcal{L}(\psi) \leqslant 0$. Since the normal component of the Reeb vector field T in M is always a Jacobi function (Lemma 3.4) we deduce in Corollary 3.6 that, if Σ does not contain vertical points (those where T is tangent to Σ), then Σ is strongly stable. Moreover, an immediate application of Theorem 3.5 yields strong stability of Σ whenever the function $|N_h|$ satisfies $\mathcal{L}(|N_h|) \leqslant 0$. It is worth mentioning that the relation between $\mathcal{L}(|N_h|)$ and the stability properties of minimal surfaces has been investigated in several works, see [13,23,14,33,17,18,20].

In Section 4 we study complete CMC surfaces with empty singular set in the model spaces $\mathbb{M}(\kappa)$ (though our construction and results can be extended to arbitrary sub-Riemannian 3-space forms). The existence of smooth stable examples in $\mathbb{M}(\kappa)$ is very restrictive due to some rigidity results that we now summarize. In the Heisenberg group $\mathbb{M}(0)$ any strongly stable minimal surface Σ with $\Sigma_0 = \emptyset$ is a vertical plane. By assuming C^2 regularity of Σ this was proved in [23,14] after some partial characterizations in [3,13]. In the