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Global solutions for the ultra-relativistic Euler equations

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A B S T R A C T

A front tracking scheme for the ultra-relativistic Euler equations is introduced. This scheme is based on piecewise constant approximations to the front tracking Riemann solutions, where continuous rarefaction waves are approximated by finite collections of discontinuities, so-called non-entropy shocks. We study the interaction estimates of the generalized shocks (entropy and nonentropy shocks) of the ultra-relativistic Euler equations and the outcoming asymptotic Riemann solution. Moreover we use a new function to measure the strengths of the waves of the ultra-relativistic Euler equations. This function has the important implication that the strength is non increasing for the interactions of the generalized shocks. This enables us to define a new kind of total variation of a solution. The main application of this scheme, is proving the global existence of weak solutions for the ultra relativistic Euler equations.

1. Introduction

This paper is concerned with the global existence of solutions to the initial value problem for the ultra-relativistic Euler equations in one space dimension,

$$\begin{aligned} &(p(3+4u^2))_t + \left(4pu\sqrt{1+u^2}\right)_x = 0,\\ &\left(4pu\sqrt{1+u^2}\right)_t + (p(1+4u^2))_x = 0, \end{aligned} \tag{1.1}$$

$$p(0,x) = p_0(x), \qquad u(0,x) = u_0(x), \tag{1.2}$$

for $x \in \mathbb{R}$ and $t \geq 0$ where $p > 0$ and $u \in \mathbb{R}$.

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A straight calculation shows that system (1.1) has eigenvalues

$$\lambda_1 = \frac{2u\sqrt{1+u^2}-\sqrt{3}}{3+2u^2} < \lambda_3 = \frac{2u\sqrt{1+u^2}+\sqrt{3}}{3+2u^2}. \tag{1.3}$$

The corresponding right eigenvectors for system (1.1) are

$$r_1 = \left(\frac{-4p}{\sqrt{3}\sqrt{1+u^2}}, 1\right)^T, \qquad r_3 = \left(\frac{4p}{\sqrt{3}\sqrt{1+u^2}}, 1\right)^T. \tag{1.4}$$

Proposition 1.1 *([4]).* *System (1.1) is strictly hyperbolic and genuinely nonlinear at each point (p,u) in the region $p>0, u\in\mathbb{R}$.*

In this paper, the front tracking scheme presented in [3] is considered as an analytical tool to solve the initial value problem for the ultra relativistic Euler equations (1.1) and (1.2). Given an initial condition $p(0,x)=p_0(x), u(0,x)=u_0(x)$ with sufficiently small total variation, we will construct an admissible weak solution (p,u), defined for all $t\geq 0$. The basic ideas involved in the construction of piecewise constant approximate solutions, based on wave-front tracking, were introduced in the papers of Dafermos [13] for scalar equations and Diperna [14] for 2×2 systems, then extended in [9,10,16] to general $N\times N$ systems with genuinely nonlinear or linearly degenerate characteristic fields. For more details about front tracking method, see [5,7,8,12,11,14,17,16] and references therein. For $n\times n$ systems the main problems in the construction of front scheme derives from the fact that the number of lines of discontinuity may approach infinity in finite time, in which case the construction would break down. For this reason, the algorithms in [9,12,14,16] are designed to get over the difficulty. Diperna [14] achieved that in case of 2×2 system, it is possible to keep the total numbers of waves finite at all times even without introducing non-physical waves. We will revisit Bressan's construction [12] and propose a further slight modification of the algorithm, still avoiding the non-physical fronts. This modification will be used in the proof of existence for the system (1.1). Namely, we use the new front tracking scheme, given in [3]. Most standard front tracking techniques [9,11,12] allow some non-physical waves, i.e. the Rankine–Hugoniot conditions are not satisfied in general. In contrast, our scheme for the ultra system gives only exact weak solutions. We will also use the strength function given in [2], which measures the strengths of the waves of the ultra-relativistic Euler equations (1.1) in a natural way. This enables us to define a new kind of total variation for the initial data of the ultra-relativistic Euler equations, for which we will prove that it is non increasing in time.

The paper is organized as follows: In Sections 2 and 3 we introduce the basic notations and review the general definition of the ultra-relativistic Euler equations, namely the parametrization of single shocks and rarefaction waves as well as the front tracking scheme. Section 4 contains the statements of the main results. In particular, Theorem 4.1 states the global existence of front tracking approximate solutions defined in time. In this section we also give a brief information about the front tracking Riemann solution (heart of the scheme). Since the front tracking technique based on the interaction of the discontinuities (generalized shocks). We study the interaction of these discontinuities. We formulate and prove the fundamental estimates for the interaction of discontinuities in Section 5. Our study of interaction of waves also allows us to determine the type of the outgoing Riemann solutions. The wavefront tracking algorithm is described in Section 6. In Section 6 we also give the proof of the main Theorem 4.1.

2. Preliminaries

In this section we review the results obtained in [15, Section 4.4] and [4], about the parametrizations of shocks and rarefaction waves, namely Lemmas 2.2 and 2.3. We consider a straight line shock $x=x(t)$ with