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# Uniqueness of minimal blow-up solutions to nonlinear Schrödinger system

Yiming Su

*School of Science, Zhejiang University of Technology, Hangzhou, PR China*

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ABSTRACT

In this paper, we are concerned with the finite time blow-up solutions to the following $N$ coupled nonlinear Schrödinger system in $\mathbb{R}^2$:

$$i\partial_t\phi_j + \Delta\phi_j + \sum_{k=1}^{N} a_{jk}|\phi_k|^2\phi_j = 0, \tag{0.1}$$

where $N \geq 2$ and the coefficient matrix $A$ satisfies $a_{jk} = a_{kj} > 0$. We study the properties of the blow-up solutions which obtain the minimal $L^2$-norm, and prove that such solution is unique, up to the symmetries of the system.


## 1. Introduction

In this paper, we consider the following $N$ coupled nonlinear Schrödinger system in $\mathbb{R}^2$:

$$\begin{cases} i\partial_t\phi_j = -\Delta\phi_j - \sum_{k=1}^{N} a_{jk}|\phi_k|^2\phi_j, & (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ \phi_j(0,x) = \phi_{j0}(x), & \phi_{j0} : \mathbb{R}^2 \to \mathbb{C},\ j = 1,2,\ldots,N, \end{cases} \tag{1.1}$$

where $N \geq 2$ and the coefficient matrix $A$ satisfies $a_{jk} = a_{kj} > 0$.

System (1.1) has received a lot of attention from mathematicians recently. It arises from many physical problems in which there are more than one component, such as in nonlinear optics and the Hartree–Fock theory for Bose–Einstein condensates. See Refs. [6,7,20,21] for the derivation and application of the system. When $N = 1$, (1.1) degenerates to the following cubic Schrödinger equation which has been wildly studied during the last decades:

$$i\partial_t\phi = -\Delta\phi - |\phi|^2\phi. \tag{1.2}$$

*E-mail address:* yimingsu@amss.ac.cn.

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And a lot of work has been devoted to establish the global well-posedness and scattering and finite time blow-up theory. See [3] for a systematic reference.

We are particularly interested in the problem in $\mathbb{R}^2$, since in this case system (1.1) is mass-critical in the sense that the scaling

$$\vec{\phi}_\lambda(t,x) = \lambda\vec{\phi}(\lambda^2 t, \lambda x) \tag{1.3}$$

leaves the mass of the solution ($L^2$-norm) invariant. Moreover, as the single equation (1.2), we can see that (1.1) possesses several symmetries: translation invariance, scaling invariance, phase invariance, Galilean invariance and pseudo-conformal invariance. Actually, if $\vec{\phi}(t,x)$ is a solution to (1.1), then for $\forall t_0, \theta, \lambda > 0 \in \mathbb{R}, x_0, \beta \in \mathbb{R}^2$, we have that $\vec{\phi}(t+t_0, x+x_0), \lambda\vec{\phi}(\lambda^2 t, \lambda x), e^{i\theta}\vec{\phi}(t,x), e^{(i\beta\cdot x - |\beta|^2 t)}\vec{\phi}(t, x-2\beta t)$ and $\frac{1}{|t|}e^{i\frac{|x|^2}{4t}}\vec{\phi}^*\left(\frac{1}{t}, \frac{x}{t}\right)$ also satisfy (1.1) respectively, where $\vec{\phi}^*$ is the complex conjugate to $\vec{\phi}$.

By applying the classic Strichartz's estimates and Kato's fixed point method [3], one can similarly establish the local well-posedness theory for Eq. (1.1) in $(H^1(\mathbb{R}^2))^N$. Namely, (1.1) admits a unique local solution $\vec{\phi}(t,x)$ in $C([0,T);(H^1(\mathbb{R}^2))^N)$ with initial data $\vec{\phi}_0(x) \in (H^1(\mathbb{R}^2))^N$ (we only consider the positive time direction for simplicity) and the following blow-up alternative holds:

either

$$T = +\infty; \tag{1.4}$$

or

$$0 < T < +\infty, \quad \text{and} \quad \lim_{t\to T^-} \sum_{j=1}^N \|\nabla\phi_j(t,x)\|_{L^2(\mathbb{R}^2)} = +\infty. \tag{1.5}$$

In the second case, $\vec{\phi}$ is called a finite time blow-up solution of (1.1) and $T$ is called the blow-up time.

Furthermore, the solution $\vec{\phi}$ obeys the following conservation laws of mass and energy:

$$M(\phi_j)(t) \triangleq \int_{\mathbb{R}^2} |\phi_j(t,x)|^2 dx = M(\phi_{j0}) = \int_{\mathbb{R}^2} |\phi_{j0}(x)|^2 dx, \tag{1.6}$$

and

$$M(\vec{\phi})(t) \triangleq \sum_{j=1}^N M(\phi_j)(t) = M(\vec{\phi}_0) = \sum_{j=1}^N M(\phi_{j0}), \tag{1.7}$$

and

$$\begin{aligned} E(\vec{\phi})(t) &\triangleq \frac{1}{2}\sum_{j=1}^N \int_{\mathbb{R}^2} |\nabla\phi_j(t,x)|^2 dx - \frac{1}{4}\sum_{j,k=1}^N \int_{\mathbb{R}^2} a_{jk}|\phi_j(t,x)|^2|\phi_k(t,x)|^2 dx \\ &= E(\vec{\phi}_0) = \frac{1}{2}\sum_{j=1}^N \int_{\mathbb{R}^2} |\nabla\phi_{j0}(x)|^2 dx - \frac{1}{4}\sum_{j,k=1}^N \int_{\mathbb{R}^2} a_{jk}|\phi_{j0}(x)|^2|\phi_{k0}(x)|^2 dx. \end{aligned} \tag{1.8}$$

Notation: throughout the paper, we denote

$$\begin{aligned} &\Omega = \{\vec{u}(x) : u_j \in H^1(\mathbb{R}^2),\ u_j \neq 0,\ \forall 1 \le j \le N\}, \\ &\Sigma = \{\vec{u}(x) : u_j \in H^1(\mathbb{R}^2), xu_j \in L^2(\mathbb{R}^2),\ \forall 1 \le j \le N\}, \\ &(L^2(\mathbb{R}^2))^N = \underbrace{L^2(\mathbb{R}^2) \times \cdots \times L^2(\mathbb{R}^2)}_{N}, \\ &(H^1(\mathbb{R}^2))^N = \underbrace{H^1(\mathbb{R}^2) \times \cdots \times H^1(\mathbb{R}^2)}_{N}, \end{aligned}$$