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On the maximal smoothing effect for multidimensional scalar conservation laws

Pierre Castelli[a], Stéphane Junca[b,*]

[a] *Lycée Audiberti, Antibes, France*
[b] *Université Côte d'Azur, Inria, CNRS, LJAD, France*

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ABSTRACT

In 1994, Lions, Perthame and Tadmor conjectured an optimal smoothing effect for entropy solutions of multidimensional scalar conservation laws. This effect estimated in fractional Sobolev spaces is linked to the flux nonlinearity. We use a new definition of a nonlinear smooth flux in order to show that the conjectured smoothing effect cannot be exceeded. First one-dimensional solutions are considered in fractional BV spaces. Then the multidimensional case is handled with a monophase solution.

1. Introduction

The following multidimensional scalar conservation laws are considered in this paper:

$$\partial_t U + \mathrm{div}_X\ F(U) = 0, \qquad U(0,X) = U_0(X) \in L^\infty(\mathbb{R}^n, \mathbb{R}). \tag{1.1}$$

A solution is said to be an entropy solution if for every convex function η,

$$\frac{\partial}{\partial t}(\eta(u)) + \frac{\partial}{\partial x}(q(u)) \leq 0 \text{ in the sense of distributions on } (0,T)\times\mathbb{R}, \text{ where } q' = \eta' f',$$

and if the initial data is recovered in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$:

$$\operatorname*{ess\,lim}_{t\to 0}\ u(x,t) = u_0(x).$$

* Corresponding author.
E-mail addresses: pierre.castelli@ac-nice.fr (P. Castelli), junca@unice.fr (S. Junca).

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The first smoothing effect measured in Sobolev spaces was obtained in 1994 by Lions, Perthame and Tadmor [23] for a flux $F \in C^1(\mathbb{R}, \mathbb{R}^n)$. It was improved by Tadmor and Tao in 2007 [29]. This smoothing effect generalizes the BV smoothing effect obtained in 1957 independently by Lax and Oleinik for a one-dimensional uniformly convex flux [22,25]. In [23] the regularity is measured in the Sobolev space $W^{s,1}_{\mathrm{loc}}(\mathbb{R}^n_X, \mathbb{R})$ with a small $s \in]0,1]$: Lions, Perthame and Tadmor conjectured that

$$s = \alpha,$$

where $\alpha \in]0,1]$ quantifies the nonlinearity – defined below – of the flux on the compact interval $K = [\inf U_0, \sup U_0]$:

Definition 1 (*Nonlinear Flux [23]*). $F : \mathbb{R} \to \mathbb{R}^n$ is said to be nonlinear on K if there exist $\alpha > 0$ and $C = C_\alpha > 0$ such that for all $\delta > 0$

$$\sup_{\tau^2+|\xi|^2=1} |W_\delta(\tau,\xi)| \le C\,\delta^\alpha,$$

where $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and $|W_\delta(\tau,\xi)|$ is the one-dimensional measure of the singular set:

$$|W_\delta(\tau,\xi)| := \{v \in K, |\tau + a(v)\cdot\xi| \le \delta\} \subset K \quad \text{and } a = F'.$$

In all the sequel, only the greatest α is considered.

In the one-dimensional case, De Lellis and Westdickenberg showed in 2003 that $s \le \alpha$ for power-law convex fluxes [15] and Jabin showed in 2010 that $s = \alpha$ for C^2 fluxes under a generalized Oleinik condition [17].

For a nonlinear multidimensional smooth flux the parameter α is determined explicitly in [20] with an equivalent definition of nonlinearity recalled in Section 2. In particular the parameter α depends on the space dimension n and satisfies: $\alpha \le \frac{1}{n}$. Moreover, Definition 4 naturally yields the construction of a supercritical family of oscillating smooth solutions – on a bounded time before shocks – exactly uniformly bounded in the optimal Sobolev space conjectured [20].

In this paper:

- we obtain an extension of the inequality $s \le \alpha$ for all nonlinear multidimensional smooth fluxes;
- we present examples of special individual solutions (and not a family of solutions as in [20]) which belong to the almost optimal Sobolev space.

In order to do so we use the fractional BV spaces which appear to be more relevant in the one-dimensional case to get the regularity and the shock structure of entropy solutions [5,14]. One-dimensional examples with low regularity given in [6,10,15] are first studied in generalized BV spaces and then extended to the multidimensional case. Notice that the construction is optimal for the one-dimensional case, at least for the class of degenerate strictly convex fluxes [5,6,15]. As in [6,9,12] these examples are not related to the convexity. We conjecture that it is also optimal for the multidimensional case, at least for fluxes smooth enough (of class C^{n+1}). For a flux only of class C^1 the natural way is to generalize the BV_Φ approach developed in [7]. The main result of the paper is about the limitation of the smoothing effect quantified with fractional derivatives. Other qualitative approaches about the structure of entropy solutions for a uniformly convex flux can be found: an SBV smoothing effect in [1,3,4] and a generic piecewise smooth local regularity in [28]. For a non uniformly convex flux the BV smoothing effect is lost [9]. Recently, a non BV solution for all time $t > 0$ has been built in [1].

Theorem 2 (*Solutions With the Minimal Sobolev Regularity Expected*). *Let $K = [m_1, m_2] \subset \mathbb{R}$ be a closed interval, $F \in C^\infty(K, \mathbb{R}^n)$ a nonlinear flux such that the associated $\alpha = \alpha[K]$ is positive. Then, for all $\varepsilon > 0$*