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Approximations of Lipschitz maps via immersions and differentiable exotic sphere theorems

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ABSTRACT

As our main theorem, we prove that a Lipschitz map from a compact Riemannian manifold M into a Riemannian manifold N admits a smooth approximation via immersions if the map has no singular points on M in the sense of F.H. Clarke, where $\dim M \leq \dim N$. As its corollary, we have that if a bi-Lipschitz homeomorphism between compact manifolds and its inverse map have no singular points in the same sense, then they are diffeomorphic. We have three applications of the main theorem: The first two of them are two differentiable sphere theorems for a pair of topological spheres including that of exotic ones. The third one is that a compact n-manifold M is a twisted sphere and there exists a bi-Lipschitz homeomorphism between M and the unit n-sphere $S^n(1)$ which is a diffeomorphism except for a single point, if M satisfies certain two conditions with respect to critical points of its distance function in the Clarke sense. Moreover, we have three corollaries from the third theorem; the first one is that for any twisted sphere Σ^n of general dimension n, there exists a bi-Lipschitz homeomorphism between Σ^n and $S^n(1)$ which is a diffeomorphism except for a single point. In particular, there exists such a map between an exotic n-sphere Σ^n of dimension $n > 4$ and $S^n(1)$; the second one is that if an exotic 4-sphere Σ^4 exists, then Σ^4 does not satisfy one of the two conditions above; the third one is that for any Grove–Shiohama type n-sphere N, there exists a bi-Lipschitz homeomorphism between N and $S^n(1)$ which is a diffeomorphism except for one of points that attain their diameters.

1. Introduction

1.1. Motivations and our main theorem

We first mention our motivations behind our purpose, which are exotic structures: No one needs the introduction of exotic n-dimensional spheres Σ^n, however Σ^n were very first discovered by Milnor [25] in

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the case of $n = 7$, which is, by definition, homeomorphic to the standard n-sphere S^n but not diffeomorphic to it. Note that the smooth 4-dimensional Poincaré conjecture (SPC4), that is, the problem of the existence of an exotic structure on the 4-sphere, is still open.

Due to exotic structures, we always have technical difficulties when investigating whether a topological sphere theorem can be reinforced into a differentiable sphere theorem. The difficulties become clearer from global Riemannian geometry's stand point: It follows from Smale's h-cobordism theorem [35] together with [34] that every homotopy sphere of dimension $n \geq 5$ is a twisted sphere, which is a smooth manifold obtained by glueing two standard n-discs along their boundaries under a boundary diffeomorphism. This implies that every Σ^n $(n > 4)$ is actually twisted, since Σ^n is a homotopy sphere. Applying Weinstein's deformation technique for metrics [37, Proposition C] to both of two discs embedded smoothly into a twisted sphere X of general dimension, we see that X admits a metric such that the cut locus[1] of some point on X is a single point. By all together above, we have

Theorem 1.1 (*Also See [2, Proposition 7.19]*)**.** *Every exotic sphere Σ^n of dimension $n > 4$ admits a metric such that there is a point whose cut locus consists of a single point.*

Thus, it is very difficult for us to notice the difference between Σ^n and S^n from the point of view of two exponential maps at the single point on Σ^n and at any point on S^n. For example, one of open problems in global Riemannian geometry is if a Grove–Shiohama type n-sphere can be diffeomorphic to S^n. Here, a complete Riemannian manifold V is called a *Grove–Shiohama type sphere* if sectional curvature $K_V \geq 1$ and its diameter $\operatorname{diam}(V) > \pi/2$ [18]. Since such a V is twisted, from Theorem 1.1 we can infer that some single cut point on V is a big obstacle whenever approximating a homeomorphism, in fact which is bi-Lipschitz, between V and S^n by diffeomorphisms.

Hence, there is a cardinal importance to do an analysis of such singular points on an arbitrary manifold. For this, we employ a notion used in non-smooth analysis of F.H. Clarke [5,6], i.e., a non-singular point for a Lipschitz function/map in this article. The following example shows that the non-smooth analysis is a strong tool in differential geometry.

Example 1.2. Let M be a complete Riemannian manifold, d the distance function of M. Take any point $p \in M$, and fix it. Set $d_p(x) := d(p, x)$ for all $x \in M$. Then, the point $q \in M \setminus \{p\}$ is a *critical point of* d_p (or *critical point for* p) in the sense of Grove–Shiohama [18], if for every nonzero tangent vector $v \in T_qM$ at q, there exists a minimal geodesic segment γ emanating from q to p such that

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}.$$

Here, $\angle(v, \dot{\gamma}(0))$ denotes the angle between v and $\dot{\gamma}(0) := (d\gamma/dt)(0)$. Note that a critical point of d_p is the cut point of p. Assume that, for some $r > 0, \partial B_r(p) := \{x \in M \mid d(p, x) = r\}$ has no critical points of d_p. By Gromov's isotopy lemma [15], $\partial B_r(p)$ is a topological submanifold of M. Since $\partial B_r(p)$ is also free of critical points of d_p in the sense of Clarke (see Example 1.9), it follows from Clarke's implicit function theorem [7] that in fact, $\partial B_r(p)$ is a *Lipschitz* submanifold of M. We are going to give the definition of the Clarke sense later (see Definition 1.5 in Section 1.2).

Our purpose of this article is *to establish an approximation method for a Lipschitz map via diffeomorphisms using the notion of non-smooth analysis and to apply this method to prove differentiable sphere theorems.* That is, our main theorem is as follows:

[1] The cut locus $\operatorname{Cut}(p)$ of a point p in a complete Riemannian manifold M is, by definition, the closure of the set of all points $x \in M$ such that there are at least two minimal geodesics emanating from p to x. Then, a point in $\operatorname{Cut}(p)$ is called the cut point of p. For example, $\operatorname{Cut}(N)$ of $N := (0, \ldots, 0, 1) \in S^n(1)$ is $\{(0, \ldots, 0, -1)\}$, where $S^n(1) := \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$. Note that the distance function from a point x of M is not differentiable at the cut point of x.