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Nonradial sign changing solutions to Lane–Emden problem in an annulus

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ABSTRACT

In this paper we prove the existence of continua of nonradial solutions for the Lane–Emden equation in the annulus. In a first result we show that there are infinitely many global continua detaching from the curve of radial solutions with any prescribed number of nodal zones. Next, using the fixed point index in cone, we produce nonradial solutions with a new type of symmetry. This result also applies to solutions with fixed signed, showing that the set of solutions to the Lane–Emden problem has a very rich and complex structure.

1. Introduction

This work deals with the Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } A, \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1.1}$$

where $A = \{x \in \mathbb{R}^N : a < |x| < b\}$ is an annulus of $\mathbb{R}^N$ with $N \geq 2$ and $p > 1$. Problem (1.1) is called subcritical for any value of p in dimension 2 and for $1 < p < \frac{N+2}{N-2}$ in dimension $N \geq 3$. In the subcritical case solutions can be found as critical points of the functional

$$F(u) := \frac{1}{2}\int_A |\nabla u|^2 - \frac{1}{p+1}\int_A |u|^{p+1} \tag{1.2}$$

on the space $H_0^1(A)$ using the compact embedding of $L^{p+1}(A)$ into $H_0^1(A)$ for $p+1 < \frac{2N}{N-2}$.

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When $N \geq 3$ and $p \geq \frac{N+2}{N-2}$ the problem is called critical or supercritical and variational methods or critical points theory can no longer be used to find solutions because the functional F is not well defined on the space $H^1_0(A)$ nor is it compact. Anyway this problem can be solved in the framework of radial solutions since the embedding of $L^{p+1}_{rad}(A)$ into $H^1_{0,rad}(A)$ is compact for every value p and any dimension N. This is indeed a consequence of the radial Lemma (see for example [15]) and problem (1.1) always admits a positive solution in the space $H^1_{0,rad}(A)$. Further sign changing solutions in $H^1_{0,rad}(A)$ can be found using the minimization procedure introduced in [5] (see also [4] where this procedure is applied to problem (1.1)).

In particular for any $m \in \mathbb{N}$, there are exactly two mutually opposite radial solutions to (1.1) which have exactly m nodal zones, see [16].

So for any $p > 1$ and $m \in \mathbb{N}$ there is only one radial solution to (1.1) which has m nodal zones and is positive in the first one (i.e. has positive derivative in $|x| = a$): we shall denote it by u^m_p and introduce the curve

$$\mathcal{S}^m := \left\{(p, u^m_p) : p \in (1, +\infty)\right\}.$$

In the following we shall regard the number of nodal zones m as fixed, therefore we shall omit the dependence on m, when it does not give rise to ambiguity.

In this paper we find nonradial solutions which spread from $\mathcal{S}$ by using bifurcation theory, a powerful tool for dealing with supercritical problems, which has already been applied to solutions with fixed-sign in [13]. A necessary condition is a change of the Morse index of u_p. The computation of the Morse index of nodal radial solutions to the Lane–Emden problem has been the subject of various studies recently. For instance in the unit ball it has been exactly computed for large values of p in [8] when $N = 2$, and for almost critical values of p in [9] when $N \geq 3$. For annular type domains, the paper [4] gives a characterization of the Morse index in terms of a related Sturm–Liouville problem and describes its asymptotic behavior as $p \to 1$ and $+\infty$. This yields, incidentally, that there are infinitely many values of p where the Morse index does change, although there still are nontrivial difficulties in deducing actual bifurcation because there is no evidence that such change is odd. Since our problem can be supercritical no variational structure can be used and only an *odd* change in the Morse index can produce a bifurcation result. Such difficulties are not present when dealing with positive solutions, because in that case only the first eigenvalue of a radial associated problem can produce degeneracy and this ensures that the related eigenfunctions of the Laplace–Beltrami operator which are $O(N-1)$-invariant span a one dimensional space. This argument is the key for the closing part of the proof of [13, Theorem 1.7] and also applies to more general (non-autonomous) problems, as in [12,1], or [2]. With this approach one can construct nonradial solutions which still are $O(N-1)$-invariant. Unfortunately it does not apply to nodal solutions, because the structure of degeneracy is much more involved, and one has to take into account the eigenfunctions of the Laplace–Beltrami operator related to many different eigenvalues: in general this could produce an even change in the Morse index. In the present paper we overcome this obstacle and present two slightly different bifurcation results. In the first one bifurcation is obtained by the classical Leray–Schauder degree approach, after showing that the change of the Morse index is actually odd in a suitable linear space which varies according to p and to the dimension N. It can be either the entire $C^{1,\gamma}_0(A)$ or a subspace of type

$$X^n := \left\{u \in C^{1,\gamma}_0(A) : \ u(r, \varphi, \theta) \text{ is } 2\pi/n \text{ periodic and even w.r.t. } \varphi\right\}. \tag{1.3}$$

Here we have denoted by (r, φ, θ) the spherical coordinates in $\mathbb{R}^N$ with $r = |x| \in [0, +\infty)$, $\varphi \in [0, 2\pi]$ and $\theta = (\theta_1, \ldots \theta_{N-2}) \in (0, \pi)^{N-2}$: