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On a cross-diffusion model for multiple species with nonlocal interaction and size exclusion

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ABSTRACT

The aim of this paper is to study a PDE model for two diffusing species interacting by local size exclusion and global attraction. This leads to a nonlinear degenerate cross-diffusion system, for which we provide a global existence result. The analysis is motivated by the formulation of the system as a formal gradient flow for an appropriate energy functional consisting of entropic terms as well as quadratic nonlocal terms. Key ingredients are entropy dissipation methods as well as the recently developed boundedness by entropy principle. Moreover, we investigate phase separation effects inherent in the cross-diffusion model by an analytical and numerical study of minimizers of the energy functional and their asymptotics to a previously studied case as the diffusivity tends to zero. Finally we briefly discuss coarsening dynamics in the system, which can be observed in numerical results and is motivated by rewriting the PDEs as a system of nonlocal Cahn–Hilliard equations.

1. Introduction

Mathematical models with local repulsion and global attraction received strong attention in the last decades, in particular motivated by applications in biology ranging from bacterial chemotaxis (cf. e.g. [17,25,47,55]) to macroscopic motion of animal groups (cf. e.g. [10,15,31,52]) as well as applications in other fields of science (cf. e.g. [62,45]). The macroscopic modelling of the density evolution leads to partial differential equations with nonlinear diffusion and an additional nonlocal term. The majority of such models can be formulated as metric gradient flows for the density ρ with some energy functional consisting of a local and a nonlocal interaction term

$$E(\rho) = \int_\Omega e(\rho) - \rho(K * \rho)\, dx, \tag{1.1}$$

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where e is a convex functional, K a nonnegative interaction kernel, and $\Omega \subset \mathbb{R}^N$. Throughout the paper, Ω may always be unbounded and we will explicitly remark when its boundedness is needed. Various results on the analysis of energy minimizers respectively stationary states (cf. [2,5,16,19,22,26,27,29,32,39,50,58,59]) and the gradient flow dynamics of the form (cf. [6,9,8,16,28,30,36,65])

$$\partial_t \rho = \nabla \cdot (M(\rho)\nabla E'(\rho)) \tag{1.2}$$

have been achieved in the last years, which led to a good understanding of such models and phenomena.

Much less is known however in the case of multi-species systems, which received most attention only recently (cf. e.g. [1,14,13,20,44,54,57,60]). With different species, the modelling leads to nonlinear degenerate cross-diffusion systems for the densities of all species, again with some nonlocal terms. The majority of work was concerned with the derivation of models including formal and computational studies, rigorous results are so far mainly available without the nonlocal interaction terms. First rigorous studies of stationary problems (cf. [24,18,33]) show interesting phase separation phenomena, whose dynamics seems rather unexplored so far. In this paper we hence study a nonlocal cross-diffusion model for two species called red (density r) and blue (density b) for simplicity, which can be derived from a lattice-based microscopic model with size exclusion (cf. [20,57]):

$$\partial_t r = \nabla \cdot (\varepsilon(1-\rho)\nabla r + \varepsilon r \nabla \rho + r(1-\rho)[\nabla(c_{11} K * r - K * b) + \nabla V]), \tag{1.3}$$

$$\partial_t b = \nabla \cdot (D\varepsilon(1-\rho)\nabla b + D\varepsilon b \nabla \rho + Db(1-\rho)[\nabla(c_{22} K * b - K * r) + \nabla V]), \tag{1.4}$$

either on the whole space or in a bounded domain supplemented with no-flux boundary conditions. The positive parameter ε regulates the strength of the diffusion relative to the nonlocal convection terms. Here K is again the interaction kernel, and the constants $c_{ii} < 0, i = 1, 2$ measure the strength of self-interaction, while the strength of the cross-interaction is scaled to unity (in accordance with the notation of [33]). The time scaling is chosen such that r has a unit diffusion coefficient, and D is the potentially different diffusion coefficient of b. The function ρ is the nonnegative total density

$$\rho = r + b, \tag{1.5}$$

naturally bounded from above by one. This system (for $V \equiv 0$) is a gradient flow for the energy functional

$$F^\varepsilon(r, b) = \varepsilon F^E(r, b) + F^0(r, b) \tag{1.6}$$

consisting of the nonlocal interaction

$$F^0(r, b) = \int_\Omega c_{11} r(K * r) - r(K * b) - b(K * r) + c_{22} b(K * b)\, dx \tag{1.7}$$

and the entropy term

$$F^E(r, b) = \int_\Omega r \log r + b \log b + (1-\rho)\log(1-\rho)\, dx. \tag{1.8}$$

For Ω unbounded we need a confining potential V and the energy is modified to

$$E^\varepsilon(r, b) = F^\varepsilon(r, b) + F^C(r, b), \qquad F^C(r, b) = \int_\Omega (r+b)V\, dx. \tag{1.9}$$

We shall from now on always use the letter F to denote the energy without confining potential and E if the potential is present. This energy is to be considered on the set of bounded densities with given mass

$$\mathcal{A} = \left\{(r, b) \in L^1(\Omega, \mathbb{R}^+)^2 : \int_{\mathbb{R}^N} r\, dx = m_r,\ \int_{\mathbb{R}^N} b\, dx = m_b, \rho = r + b \le 1 \text{ for a.e. } x \in \Omega\right\}, \tag{1.10}$$

which can be shown to be invariant under the dynamics of (1.3), (1.4). The special case of minimizing F^0 on $\mathcal{A}$ was recently investigated by Cicalese et al. [33] and it appears obvious that F^ε is the natural entropic version and (1.3), (1.4) the natural dynamic model leading to such a minimization problem in the large time asymptotics.

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