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Stabilization in a higher-dimensional quasilinear Keller–Segel system with exponentially decaying diffusivity and subcritical sensitivity

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ABSTRACT

The quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{$\star$}$$

is considered under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with smooth boundary, where the focus is on cases when herein the diffusivity $D(s)$ decays exponentially as $s \to \infty$.

It is shown that under the subcriticality condition that

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \qquad \text{for all } s \geq 0 \tag{0.1}$$

with some $C > 0$ and $\alpha < \frac{2}{n}$, for all suitably regular initial data satisfying an essentially explicit smallness assumption on the total mass $\int_\Omega u_0$, the corresponding Neumann initial–boundary value problem for ($\star$) possesses a globally defined bounded classical solution which moreover approaches a spatially homogeneous steady state in the large time limit. Viewed as a complement of known results on the existence of small-mass blow-up solutions in cases when in (0.1) the reverse inequality holds with some $\alpha > \frac{2}{n}$, this confirms criticality of the exponent $\alpha = \frac{2}{n}$ in (0.1) with regard to the singularity formation also for arbitrary $n \geq 2$, thereby generalizing a recent result on unconditional global boundedness in the two-dimensional situation.

As a by-product of our analysis, without any restriction on the initial data, we obtain boundedness and stabilization of solutions to a so-called volume-filling chemotaxis system involving jump probability functions which decay at sufficiently large exponential rates.

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1. Introduction

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the quasilinear parabolic problem

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

which is used in mathematical biology to describe the evolution of bacterial populations, at density denoted by $u = u(x,t)$, in response to a chemical signal, at concentration $v = v(x,t)$, produced by themselves. In generalization of the classical Keller–Segel chemotaxis system obtained upon the particular choices

$$D \equiv 1 \quad \text{and} \quad S(s) = s, \quad s \geq 0, \tag{1.2}$$

the model (1.1) may account for various types of nonlinear diffusion and cross-diffusion mechanisms, where especially saturation effects at large cell densities appear to play a predominant role in refined modeling approaches [9,17]; accordingly, in such contexts it will be of particular interest to determine how far choices of D and S substantially below those in (1.2) may still lead to singularity formation in (1.1), as known to occur for large classes of initial data in the Keller–Segel system (1.1)–(1.2) when either $n \geq 3$ [15], or $n = 2$ and the total mass of cells is suitably large [6,8].

In this respect, previous results indicate that a certain dimension-dependent power-type asymptotic behavior of the ratio $\frac{S(s)}{D(s)}$ for large values of s should be critical: It is known, for instance, that if D and S are sufficiently smooth functions on $[0,\infty)$ such that $D > 0$ on $(0,\infty)$ and

$$\liminf_{s\to\infty} \frac{s\left(\frac{S}{D}\right)'(s)}{\left(\frac{S}{D}\right)(s)} > \frac{2}{n}, \tag{1.3}$$

then still some solutions to (1.1) exist which blow up either in finite or infinite time [14,2,3,7]. On the other hand, any such unboundedness phenomenon is entirely ruled out if with some $\varepsilon > 0$ and $C > 0$ we have

$$\frac{S(s)}{D(s)} \leq Cs^{\frac{2}{n}-\varepsilon} \quad \text{for all } s \geq 1, \tag{1.4}$$

and if in addition D decays at most algebraically in the sense that

$$\liminf_{s\to\infty} \Big(s^p D(s)\Big) > 0 \tag{1.5}$$

for some $p > 0$ ([13]; cf. also [11,7] for some precedents).

In cases when D decays substantially faster, however, the literature apparently provides only quite few rigorous results on (1.1) for subcritical behavior of $\frac{S}{D}$ in the sense that e.g. (1.4) holds; this may be viewed as reflecting the circumstance that straightforward adaptations of standard regularity techniques, based e.g. on iterative arguments of Moser or DeGiorgi type, seem inappropriate in such situations. Correspondingly, the only results available so far seem to concentrate on the mere questions of global solvability without asserting boundedness [1], [2, Theorem 1.6], [16], or are restricted to the particular spatially two-dimensional setting with exponentially decaying D, in which the Moser–Trudinger inequality can be used to firstly derive global bounds for e^u in $L^p(\Omega)$ for some $p > 0$, from which global boundedness of arbitrary classical solutions can be obtained by means of an iterative argument [4].

It is the goal of the present work to provide some further rigorous evidence indicating that also in higher-dimensional situations, the power-type asymptotic behavior $\frac{S(s)}{D(s)} \simeq s^{\frac{2}{n}}$ indeed is critical regarding the global

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