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Entropic structure and duality for multiple species cross-diffusion systems

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ABSTRACT

This paper deals with the existence of global weak solutions for a wide class of (multiple species) cross-diffusions systems. The existence is based on two different ingredients: an entropy estimate giving some gradient control and a duality estimate that gives naturally L^2 control. The heart of our proof is a semi-implicit scheme tailored for cross-diffusion systems firstly defined in Desvillettes et al. (2015) and a (nonlinear Aubin–Lions type) compactness result developed in Moussa (2016) and Andreianov et al. (2015) that turns the (potentially weak) gradient estimates into almost everywhere convergence. We apply our results to models having an entropy relying on the *detailed balance condition* exhibited by Chen *et. al.* in Chen et al. (2016).

1. Introduction

In 1979, Shigesada, Kawasaki and Teramoto introduced in [27] the following system (that we denote SKT), on $Q_T := [0,T] \times \Omega$ where $\Omega \subset \mathbb{R}^d$ is some regular bounded open set

$$\begin{cases} \partial_t u_1 - \Delta\Big[(d_1 + a_{11}u_1 + a_{12}u_2)u_1\Big] = u_1(\rho_1 - s_{11}u_1 - s_{12}u_2), \\ \partial_t u_2 - \Delta\Big[(d_2 + a_{22}u_2 + a_{21}u_1)u_2\Big] = u_2(\rho_2 - s_{21}u_1 - s_{12}u_2). \end{cases}$$

The latter aims at describing the behavior of two populations (through their density functions $u_1, u_2 \geq 0$) involving different mechanisms: self-diffusion (a_{11}, a_{22} terms), cross-diffusion (a_{12}, a_{21} terms) and growth terms, modeling reproduction (ρ_1, ρ_2 terms) or competition (s_{ij} terms). The existence theory for the corresponding Cauchy boundary value problem is a rich saga. As far as classical solutions are concerned, the cornerstone of the theory is Amman theorem [1–3] which ensures local existence of solutions and gives

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also a criterion to check for possible global solution (which amounts to control some Sobolev norms of the solution). As a matter of fact, up to now classical global solutions are only known to exist under strong assumptions on the coefficients, for instance in the case of a weak coupling in the diffusion matrix ($a_{21} = 0$ above) known as the triangular case (see [20,21] for instance or [14] for a recent result) or equal diffusion rates without self-diffusion, see [19,24]. Concerning weak solutions, it is a striking fact that until the beginning of the 2000's, no global solutions were known for the (full) SKT model. The breakthrough occurred in several steps by Jüngel et al. in [13,8,9], the core of the construction being the discovery of the following entropy for the system

$$\mathcal{E}(t) := \int_\Omega a_{21}(u_1 \log u_1 - u_1 + 1) + a_{12}(u_2 \log u_2 - u_2 + 1).$$

As detailed in [9], this convex functional is controlled over the time through

$$\mathcal{E}'(t) + \mathcal{D}(t) \le C(1 + \mathcal{E}(t)),$$

where $\mathcal{D}(t)$ is some dissipative (nonnegative) term. The entropy control leads to $u_i \log u_i \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$ and one can also recover (at least) from the dissipative term that $\nabla\sqrt{u_i} \in \mathrm{L}^2(Q_T)$. The previous *a priori* estimates pave the way to the (strong) convergence of adequate approximating procedures. This is not specific to the SKT model and has been transposed since 2006 to several variations of this system, see Section 1.1 for more details.

This manuscript is devoted to an existence theorem for generalizations of the SKT model which take the form

$$\partial_t u_i - \Delta(p_i(U)u_i) = r_i(U)u_i, \tag{1}$$

where $1 \le i \le I$, $U = (u_i)_{1\le i\le I}$,

$$p_i \in \mathscr{C}^0(\mathbb{R}_+^I, \mathbb{R}_+) \cap \mathscr{C}^1((\mathbb{R}_+^*)^I, \mathbb{R}_+), \tag{2}$$

and the reaction terms are continuous functions on $\mathbb{R}_+^I$ that can have for instance form

$$r_i(U) = \rho_i - \sum_{j=1}^I c_{ij} u_j^{\alpha_{ij}}, \quad \rho_i, c_{ij}, \alpha_{ij} \ge 0,\ \alpha_{ij} < 1. \tag{3}$$

Introducing $A(U) := (p_i(U)u_i)_{1\le i\le I}$ and $R(U) := (r_i(U)u_i)_{1\le i\le I}$, the set of scalar equations (1) is equivalent to the vectorial one

$$\partial_t U - \Delta\big[A(U)\big] = R(U). \tag{4}$$

We will use both formulations in the sequel, keeping in mind that capital letters refer to vectors and lowercase to scalars.

Notations: In all what follows, we denote by $Q_T = (0,T)\times\Omega$ the parabolic cylinder. The space $\mathrm{H}^{-1}(\Omega)$ is the dual of the elements of $\mathrm{H}^1(\Omega)$ having 0 average, an element $u \in \mathrm{H}^{-1}(\Omega)$ is thus characterized by

$$\forall\varphi \in \mathrm{H}^1(\Omega), \quad \int_\Omega (\varphi - \overline{\varphi})u \le \|\varphi\|_{\mathrm{H}^1(\Omega)}\|u\|_{\mathrm{H}^{-1}(\Omega)},$$

where $\overline{\varphi}$ is the average of φ on Ω. For two vectors $X = (x_i)_i$ and $Y = (y_i)_i$ of $\mathbb{R}^I$ we write $X \le Y$ if and only if the inequality is satisfied for each of their components, and use the same convention for $<$. The tensor $X : Y$ is the square-matrix $(x_i y_j)_{i,j}$. Finally, if U is a vector-valued or matrix-valued function defined on Q_T and E is some vector space of (scalar) functions defined on Q_T, we write simply $U \in \mathrm{E}$ to specify that each components of U belongs to E.