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Existence of solutions for a class of fourth order cross-diffusion systems of gradient flow type[☆]

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ABSTRACT

This article is concerned with the existence and the long time behavior of weak solutions to certain coupled systems of fourth-order degenerate parabolic equations of gradient flow type. The underlying metric is a Wasserstein-like transportation distance for vector-valued functions, with nonlinear mobilities in each component. Under the hypothesis of (flat) convexity of the driving free energy functional, weak solutions are constructed by means of the variational minimizing movement scheme for metric gradient flows. The essential regularity estimates are derived by variational methods.

1. Introduction

In this article, we study the existence and the large-time behavior of vector valued solutions $u : \mathbb{R}_{\geq 0} \times \mathbb{R} \to S \subset \mathbb{R}^n$ to the following coupled system of nonlinear fourth-order equations in one spatial dimension:

$$\partial_t u + \partial_x \left(\mathbf{M}(u) \left[\partial_x^2 \nabla_p f(\partial_x u, u) - \partial_x \nabla_z f(\partial_x u, u) \right] \right) = 0, \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}$, subject to the initial condition $u(0, \cdot) = u^0$ for a given function $u^0 : \mathbb{R} \to S$. We refer to the cuboid

$$S = [S_1^\ell, S_1^r] \times \cdots \times [S_n^\ell, S_n^r] \subset \mathbb{R}^n \tag{1.2}$$

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with given lower-left and upper-right corners $S^\ell, S^r \in \mathbb{R}^n$, respectively, as *value space*. The *mobility matrix* $\mathbf{M}(\cdot) : S \to \mathbb{R}^{n\times n}$ is assumed to be *fully decoupled* in the sense of [37], that is:

$$\mathbf{M}(z) = \begin{pmatrix} \mathbf{m}_1(z_1) & & & \\ & \mathbf{m}_2(z_2) & & \\ & & \ddots & \\ & & & \mathbf{m}_n(z_n) \end{pmatrix}, \tag{1.3}$$

with n nonnegative and concave (scalar) mobility functions $\mathbf{m}_k : [S_k^\ell, S_k^r] \to \mathbb{R}$. For the precise assumptions on $\mathbf{M}$, see Definition 2.1. Finally, $f : \mathbb{R}^n \times S \to \mathbb{R}$ is a prescribed smooth *free energy density* $f = f(p, z)$, with properties that are specified in Assumption 2.2. Notice that the gradients ∇_z and ∇_p in (1.1) act with respect to the n components of u and of $\partial_x u$, respectively.

Introducing the gradient-dependent free energy functional

$$\mathcal{E}(u) = \int_{\mathbb{R}} f(\partial_x u, u)\,\mathrm{d}x, \tag{1.4}$$

it becomes apparent that (1.1) is – formally – a gradient flow in the potential landscape of $\mathcal{E}$,

$$\partial_t u = \partial_x \left[\mathbf{M}(u)\,\partial_x \frac{\delta\mathcal{E}}{\delta u}(u)\right]. \tag{1.5}$$

The main purpose of the paper at hand is to carry out a new approach – alternative to the already existing ones – to prove the existence of weak solutions to equations of type (1.1) and to study their long time behavior on the basis of the indicated gradient flow structure, using variational methods. The key to fit (1.5) into the theory of gradient flows in metric spaces is the recent results [37] concerning the induced metrics $\mathbf{W_M}$ on the space $\mathscr{M}(\mathbb{R}; S)$ of S-valued measurable functions. A brief review of some essential properties of $\mathbf{W_M}$ is provided in Section 3.2. Gradient flows in $\mathbf{W_M}$ for simpler functionals than (1.4) have been studied in [37], following up on the results [10,24,7] for scalar equations with nonlinear mobility functions.

The initial motivation for studying (1.1) is its similarity to multi-component Cahn–Hilliard systems (see e.g. [15] for a review on their derivation), which are of the general form

$$\partial_t u - \partial_x\big(\widetilde{\mathbf{M}}(u)\partial_x\mu(u)\big) = 0, \qquad \mu(u) = -\varGamma\partial_x^2 u + \nabla_z\varPsi(u). \tag{1.6}$$

Here $\varGamma \in \mathbb{R}^{n\times n}$ is a positive definite matrix, $\varPsi : \widetilde{S} \to \mathbb{R}$ is the homogeneous part of the free energy density, and $\widetilde{\mathbf{M}}(\cdot) : \widetilde{S} \to \mathbb{R}^{n\times n}$ is usually referred to as Onsager matrix. In the notations above, the (total) free energy density would be given by

$$\widetilde{f}(p, z) = \frac{1}{2}p^{\mathrm{T}}\varGamma p + \varPsi(z). \tag{1.7}$$

In (1.6), a typical choice for the value space $\widetilde{S} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional Gibbs-simplex (the entries of $z \in \widetilde{S}$ are non-negative and sum up to one), and the adapted Onsager matrix of Maxwell–Stefan type is given by $\widetilde{\mathbf{M}}_{\mathrm{MS}}(z) = \mathrm{diag}(z) - zz^{\mathrm{T}}$. Properties of solutions to (1.6) have been studied by various authors, and the first rigorous existence result has been given in [11].

We present our method in the formally most transparent setting in which the value space is given by the cuboid (1.2), and the mobility matrix has the diagonal form (1.3). The *cross diffusion* between the n species is thus induced by means of the free energy $\mathcal{E}$ — e.g., by choosing $\varGamma$ non-diagonal in (1.7) — but not by means of the mobility matrix $\mathbf{M}$. For Cahn–Hilliard systems (1.6) with other specific choices of the mobility matrix (like $\widetilde{\mathbf{M}}_{\mathrm{MS}}$ above), the same variational approach still leads to a well-defined and weakly convergent semi-discrete approximation of the gradient flow, however, our method for identifying the limit curve as a weak solution to the PDE might fail.