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On the structure of continua with finite length and Gołąb's semicontinuity theorem

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ABSTRACT

The main results in this note concern the characterization of the length of continua[1] (Theorem 2.5) and the parametrization of continua with finite length (Theorem 4.4). Using these results we give two independent and relatively elementary proofs of Gołąb's semicontinuity theorem.

1. Introduction

Let $\mathscr{F}$ be the class of all continua K contained in $\mathbb{R}^d$, endowed with the Hausdorff distance. A classical result due to S. Gołąb (see [8, Section 3], or [6, Theorem 3.18]) states that the length, that is, the function $K \mapsto \mathscr{H}^1(K)$, is lower semicontinuous on $\mathscr{F}$. Variants of this semicontinuity result, together with well-known compactness properties of $\mathscr{F}$, play a key role in the proofs of several existence results in the Calculus of Variations, from optimal networks [9] to image segmentation [2] and quasi-static evolution of fractures [3]. In particular, Gołąb's theorem has been extended to general metric spaces in [1, Theorem 4.4.17], and [9, Theorem 3.3].[2]

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1 As usual, a *continuum* is a connected compact metric space (or subset of a metric space), and *length* stands for the one-dimensional Hausdorff measure $\mathscr{H}^1$.

2 The proof in [1] is actually incomplete; the missing steps were given in [9].

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It should be noted that none of the proofs of Gołąb's theorem mentioned above is completely elementary. On the other hand, the counterpart of this result for paths, namely that the length of a path $\gamma : [0,1] \to X$ is lower semicontinuous with respect to the pointwise convergence of paths, is elementary and almost trivial. This sharp contrast is due to the fact that the definitions of length of a path and of one-dimensional Hausdorff measure of a set are utterly different, even though they aim to describe (essentially) the same geometric quantity. More precisely, the length of a path, being defined as a supremum of finite sums which are clearly continuous, is naturally lower semicontinuous, while the definition of Hausdorff measure is based on Caratheodory's construction, and is designed to achieve σ-subadditivity, not semicontinuity.

In this note we point out a couple of relations/similarities between the one-dimensional Hausdorff measure of continua and the length, which we then use to give two independent (and relatively elementary) proofs of Gołąb's theorem. We think, however, that these results are interesting in their own right.

Firstly, in Theorem 2.5 we show that for every continuum X there holds

$$\mathscr{H}^1(X) = \sup\Big\{ \sum_i \operatorname{diam}(E_i) \Big\},$$

where the supremum is taken over all finite families $\{E_i\}$ of disjoint *connected* subsets of X. (Note the resemblance with the definition of length of a path.)

Secondly, in Theorem 4.4 we show that every continuum X with finite length admits a sort of canonical parametrization; more precisely, there exists a path $\gamma : I \to X$ with length equal $2\,\mathscr{H}^1(X)$ which "goes through almost every point of X twice, once moving in a direction, and once moving in the opposite direction", the precise statement requires some technical definitions and is postponed to Section 4.

This paper is organized as follows: Sections 2 and 4 contain the two results mentioned above (Theorems 2.5 and 4.4) and the corresponding proofs of Gołąb's theorem. Section 3 contains a review of some basic facts about paths with finite length in a metric space which are used in Section 4, and can be skipped by the expert reader. This review is self-contained and limited in scope; a more detailed presentation of the theory of paths with finite length in metric spaces can be found in [1, Chapter 4], while continua with finite length have been studied in detail in [4] (see also [7]).

Since the results described in this paper are rather elementary (in particular Theorem 2.5), we strove to keep the exposition self-contained, and avoid in particular the use of advanced results from Geometric Measure Theory. On the other hand, proofs are sometimes just sketched, with all steps clearly indicated but many details left to the reader.

2. A characterization of length

The main results in this section are the characterizations of the length of sets with countably many connected components (and in particular of continua) given in Theorem 2.5 and Proposition 2.8. Using the former result we give our first proof of Gołąb's theorem (Theorem 2.9).

2.1. Notation. Through this paper X is a metric space endowed with the distance d. Given $x \in X$ and E, E' subsets of X we set:

$B(x,r)$	*closed* ball with center x and radius $r > 0$;
$\operatorname{diam}(E)$	diameter of E, i.e., $\sup\{d(x,x') : x, x' \in E\}$;
$\operatorname{dist}(x,E)$	distance between x and E, i.e., $\inf\{d(x,x') : x' \in E\}$;
$\operatorname{dist}(E,E')$	distance between E and E', i.e., $\inf\{d(x,x') : x \in E,\ x' \in E'\}$;
$d_H(E,E')$	Hausdorff distance between E and E', i.e., the minimum of all $r \geq 0$ such that $\operatorname{dist}(x,E') \leq r$ for every $x \in E$ and $\operatorname{dist}(x',E) \leq r$ for every $x' \in E'$;