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# Perimeter as relaxed Minkowski content in metric measure spaces

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To our dear friend Nicola, with admiration

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ABSTRACT

In this note we prove that on general metric measure spaces the perimeter is equal to the relaxation of the Minkowski contents w.r.t. convergence in measure.


## 1. Introduction

In a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, the upper and lower Minkowski contents are respectively defined by

$$\mathcal{M}_-(A) := \liminf_{r\downarrow 0} \frac{\mathfrak{m}(A^r) - \mathfrak{m}(A)}{r}, \qquad \mathcal{M}_+(A) := \limsup_{r\downarrow 0} \frac{\mathfrak{m}(A^r) - \mathfrak{m}(A)}{r}$$

for Borel sets $A$ with finite $\mathfrak{m}$-measure. The Minkowski contents, even in their non-infinitesimal versions, appear in many areas, as in the theory of concentration of measures and isoperimetric inequalities (see for instance [21,12,13] and the references therein) and the theory of random closed sets [4].

For sufficiently nice metric measure structures, the relations between Minkowski content and perimeter are well-known, see for instance Section 2.13 in [1] and Section 14.2 in [9]. Aim of this note is the investigation of more precise relations between the Minkowski content and the perimeter, as defined in the theory of $BV$ functions in metric measure spaces. In particular we prove in Lemma 2.1 and Theorem 3.6 that the lower semicontinuous envelope w.r.t. $L^1(X, \mathfrak{m})$ convergence of the Minkowski contents $\mathcal{M}_\pm(A)$ is equal to the perimeter $\mathrm{Per}(A)$. As a byproduct, we can prove that in metric measure spaces with finite $\mathfrak{m}$-measure

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the Cheeger constant

$$\gamma := \inf\left\{\frac{\mathcal{M}_+(A)}{\mathfrak{m}(A)} :\ 0 < \mathfrak{m}(A) \le \frac{\mathfrak{m}(X)}{2}\right\}$$

can be equivalently defined replacing $\mathcal{M}_+(A)$ with $\mathcal{M}_-(A)$, or with $\mathrm{Per}(A)$.

Another consequence of our result is that whenever one wants to establish, on a given space, an isoperimetric inequality of the form

$$\mathfrak{m}(A) \le f(\mathrm{Per}(A))$$

for some continuous non-decreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$, it is sufficient to prove the easier

$$\mathfrak{m}(A) \le f(\mathcal{M}_+(A)).$$

We remark that not only it is easily seen that $\mathcal{M}_+(A) \ge \mathrm{Per}(A)$ (see also the proof of Theorem 3.6), but also that the Minkowski content is a quantity that by nature is sometimes handled better than the perimeter in estimates involving the geometry of the space. An example in this direction is the recent paper [12] by Cavalletti–Mondino, which motivated our study (see also the work in progress [11], which contains results similar to ours, under curvature assumptions).

Another goal of the paper is a closer investigation of the coarea formula and of the "generic" properties of superlevel sets of Lipschitz functions. In Euclidean and other nice spaces, the combination of the Fleming–Rishel formula (involving the perimeter of superlevel sets) with the coarea formula for Lipschitz maps (involving the Hausdorff measure of level sets) provides many useful informations, even on the level sets, as illustrated in Remark 4.1. Under a suitable regularity assumption (4.2) on the metric measure structure, fulfilled in all spaces $RCD(K,\infty)$ of [7], we provide in Proposition 4.2 a metric counterpart of this, involving the Minkowski contents. Finally, we are able to make a more detailed analysis for level sets of distance functions and we conclude the paper pointing out a few open questions.

## 2. Basic setting and preliminaries

Throughout this paper $(X, \mathsf{d})$ is a metric space and $\mathfrak{m}$ is a nonnegative and $\sigma$-additive measure on its Borel $\sigma$-algebra; we always assume that $\mathfrak{m}$ is finite on bounded Borel sets. In particular, $\mathfrak{m}$ is $\sigma$-finite.

In the metric space $(X, \mathsf{d})$ we denote by $B_r(x)$ the open ball with center $x$ and radius $r$. We denote by $\mathrm{Lip}(f)$ the Lipschitz constant of a Lipschitz function $f : X \to \mathbb{R}$ and we will often use the distance function

$$\mathsf{d}_A(x) := \inf_{y\in A} \mathsf{d}(x, y)$$

from a nonempty set $A$, whose Lipschitz constant is less than 1. The slope $\mathrm{lip}(f)$ (also called local Lipschitz constant) of $f : X \to \mathbb{R}$ is defined by

$$\mathrm{lip}(f)(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)},$$

with the convention $\mathrm{lip}(f)(x) = 0$ if $x$ is an isolated point.

We denote by $\chi_A : X \to \{0, 1\}$ the characteristic function of a set A and we say that $A_h \to A$ in $\mathfrak{m}$-measure if $\int_X |\chi_{A_h} - \chi_A|\, d\mathfrak{m} \to 0$ (equivalently, $\mathfrak{m}(A_h \Delta A) \to 0$). For any nonempty set $A \subset X$ and any $r > 0$ we define the open $r$-enlargement $A^r$ of $A$ by

$$A^r := \{x \in X :\ \mathsf{d}_A(x) < r\}.$$

Notice that $A^r = (\overline{A})^r$, and that the triangle inequality gives the semigroup inclusion

$$(A^s)^t \subset A^{s+t} \quad s, t > 0. \tag{2.1}$$