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On coercive variational integrals

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ABSTRACT

It is well-known that sequential weak lower semicontinuity of a variational integral

$$\mathfrak{F}(u,\Omega)=\int_{\Omega}F(\nabla u(x))\,\mathrm{d}x$$

on the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ under a p–growth condition on the integrand F is equivalent to quasiconvexity in the sense of Morrey. We show that coercivity on Dirichlet classes likewise is equivalent to a quasiconvexity condition. We also discuss some examples and extend a sequential weak lower semicontinuity result to the case of signed integrands.

1. Introduction

Let $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be a continuous integrand satisfying for an exponent $p\in[1,\infty)$ and constant $k>0$ the growth condition

$$|F(z)|\leq k\big(|z|^p+1\big) \tag{1.1}$$

for all matrices $z\in\mathbb{R}^{N\times n}$. We consider the corresponding variational integral

$$\mathfrak{F}(u,\Omega)=\int_{\Omega}F(\nabla u(x))\,\mathrm{d}x \tag{1.2}$$

defined for Sobolev mappings $u\in\mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ and bounded open subsets $\Omega\subset\mathbb{R}^n$. For a mapping $g\in\mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ and a non-empty bounded open subset $\Omega\subset\mathbb{R}^n$ we consider $\mathfrak{F}(\cdot,\Omega)$ on the Dirichlet class

$$\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)=\big\{g+\varphi:\ \varphi\in\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)\big\}. \tag{1.3}$$

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We emphasize that this definition of Dirichlet class differs somewhat from the usual definition, but it is convenient for our purposes here, besides it is equivalent to the standard definition whenever Ω is a $\mathrm{W}^{1,p}$ extension domain (see [26]). It is by now well-known (see [13,29,27] and compare also [7, Remark 8.5(iii)] and [1,36]) that $\mathfrak{F}(\cdot,\Omega)$ is sequentially weakly lower semicontinuous on $\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$ if and only if F is quasiconvex (see Section 2 for notation and terminology). We shall generalize this semicontinuity result in the spirit of [25], see Theorem 5.1 in Section 5.

Fix an exponent $q \in [1,p]$. We say that $\mathfrak{F}(\cdot,\Omega)$ is L^q coercive on $\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$ if

$$\mathfrak{F}(u,\Omega) \to \infty \quad \text{as } \|\nabla u\|_{q,\Omega} \to \infty \text{ through } u \in \mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N). \tag{1.4}$$

We use this terminology also for $q = 1$, which is not standard but is convenient here. The notion of L^q coercivity turns out to be a property of the integrand F and thus is independent of both Ω and g, see Proposition 3.1.

There are various ways in which one can ensure that (1.4) holds. The most obvious one is to require that F satisfies a pointwise q-coercivity condition: there exist constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that $F(z) \geq c_1|z|^q + c_2$ holds for all $z \in \mathbb{R}^{N\times n}$. This however is unnecessarily restrictive in the multi-dimensional vectorial case n, $N \geq 2$ and not satisfied in many interesting cases (see [4,14,15] and also Proposition 2.1 and the examples given below). Instead a more natural condition ensuring (1.4) is that of L^q mean coercivity: there exist constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that

$$\mathfrak{F}(u,\Omega) \geq c_1\|\nabla u\|^q_{q,\Omega} + c_2 \tag{1.5}$$

for all $u \in \mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$. It might be a little surprising that the two conditions (1.4) and (1.5) are in fact equivalent under our assumptions. This and the fact that they in turn are equivalent to a quasiconvexity condition is our main result:

Theorem 1.1. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous integrand satisfying for an exponent $p \in [1,\infty)$ and constant $k > 0$ the growth condition (1.1). Then for an exponent $q \in [1,p]$ the following five statements are mutually equivalent:*

(i) *F is L^q coercive: the condition (1.4) holds for any choice of non-empty, bounded open subset $\Omega \subset \mathbb{R}^n$ and any boundary datum $g \in \mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$.*
(ii) *There exist a non-empty bounded open subset $\Omega \subset \mathbb{R}^n$ and a boundary datum $g \in \mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ such that condition (1.4) holds.*
(iii) *F is L^q mean coercive: the condition (1.5) holds for any choice of non-empty, bounded open subset $\Omega \subset \mathbb{R}^n$ and any boundary datum $g \in \mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$.*
(iv) *There exist a non-empty bounded open subset $\Omega \subset \mathbb{R}^n$ and a boundary datum $g \in \mathrm{W}^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ such that condition (1.5) holds.*
(v) *There exist a constant $c > 0$ and a matrix $z_0 \in \mathbb{R}^{N\times n}$ such that the integrand $z \mapsto F(z) - c|z|^q$ is quasiconvex at z_0.*

We present the elementary proof of Theorem 1.1 in Section 3.

There are several interesting examples of integrands that are L^p coercive in the literature. The first systematic exposition on coercive integrands more general than quadratic forms seems to be [15], see also [14,18]. We should also mention [32] that found a necessary and sufficient condition for certain functionals satisfying a monotonicity condition. The functionals considered in the present paper will in general not satisfy this monotonicity condition. In Section 4 we discuss some classes of examples and confine attention to the integrands that are positively p-homogeneous. By this we mean integrands $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying $F(tz) = t^pF(z)$ for all $z \in \mathbb{R}^{N\times n}$ and $t \geq 0$. If we strengthen the condition to the requirement that