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Existence and almost everywhere regularity of isoperimetric clusters for fractional perimeters

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The existence of minimizers in the fractional isoperimetric problem with multiple volume constraints is proved, together with a partial regularity result.

1. Introduction

The goal of this paper is establishing basic existence and partial regularity results for the fractional isoperimetric problem with multiple volume constraints. If $E \subset \mathbb{R}^n, n \geq 2$, and $s \in (0,1)$, then the *fractional perimeter of order s of E* is defined as

$$P_s(E) = \int_{\mathbb{R}^n} w_E(x)\,dx = \int_E dx \int_{E^c} \frac{dy}{|x-y|^{n+s}}. \tag{1.1}$$

The kernel $z \mapsto |z|^{-n-s}$ is not integrable near the origin, and the potential

$$w_E(x) := 1_E(x) \int_{E^c} \frac{dy}{|x-y|^{n+s}} \quad x \in \mathbb{R}^n$$

diverges like $\mathrm{dist}(x, \partial E)^{-s}$ as $x \in E$ approaches ∂E. Since t^{-s} is integrable near 0, by decomposing the integral of w_E on a small layer around ∂E as the integral along the normal rays $t \mapsto p - t\,\nu_E(p)$, $p \in \partial E$, then we see that $P_s(E)$, at leading order, is measuring the perimeter $P(E) = \mathcal{H}^{n-1}(\partial E)$ of E. This idea is made precise by the fact that, as $s \to 1^-$, $(1-s)\,P_s(E) \to c(n)\,P(E)$ for every set of finite perimeter E, see [4,14], and $(1-s)\,P_s \to c(n)P$ in the sense of Γ-convergence [31,2].

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The last few years have seen a great effort by many authors towards the understanding of geometric variational problems in the fractional setting. This line of research has been initiated in [7] with the regularity theory for the fractional Plateau's problem (see [34,20,3,9,8] for further developments in this direction), while fractional isoperimetric problems have been the subject of [22,26,27,18,19]. Examples of singular fractional minimal boundaries (boundaries with vanishing fractional mean curvature) are found in [16,17]. Boundaries with constant fractional mean curvature have also been investigated in some detail [15,5,6,12] and their study illustrates how nonlocality brings into play both complications (need for new arguments, for example when in the local case one exploits some direct ODE argument) and simplifications (because of additional rigidities): compare, for example, the stability results from [13] with those in [12].

Our goal is starting the study, in the fractional setting, of another classical geometric variational problem, namely the isoperimetric problem with multiple volume constraints. Given $N \in \mathbb{N}$, a N-cluster (or simply a cluster) $\mathcal{E}$ is a family $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ of disjoint Borel subsets of $\mathbb{R}^n$. The sets $\mathcal{E}(h), h = 1, \dots, N$, are called the chambers of $\mathcal{E}$, while $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ is called the exterior chamber of $\mathcal{E}$. When $|\mathcal{E}(h)| < \infty$ for $h = 1, \dots, N$, then the volume vector $m(\mathcal{E})$ of $\mathcal{E}$ is defined as

$$m(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|) \in \mathbb{R}^N$$

while the fractional s-perimeter of $\mathcal{E}$ is given by

$$P_s(\mathcal{E}) = \frac{1}{2} \sum_{h=0}^{N} P_s(\mathcal{E}(h)). \tag{1.2}$$

Given $m \in \mathbb{R}_+^N$ (that is, $m_h > 0$ for $h = 1, \dots, N$), we consider the following isoperimetric problem

$$\inf \Big\{ P_s(\mathcal{E}) : m(\mathcal{E}) = m \Big\}. \tag{1.3}$$

Every minimizer in (1.3) is called an isoperimetric cluster. The following theorem is our main result.

Theorem 1.1. *For every $m \in \mathbb{R}_+^N$ there exists an isoperimetric cluster $\mathcal{E}$ with $m(\mathcal{E}) = m$. If we set*

$$\partial\mathcal{E} = \Big\{ x \in \mathbb{R}^n : \exists h = 1, \dots, N \text{ such that } 0 < |\mathcal{E}(h) \cap B_r(x)| < |B_r(x)| \ \forall r > 0 \Big\} \tag{1.4}$$

then $\partial\mathcal{E}$ is bounded and there exists a closed set $\Sigma(\mathcal{E}) \subset \partial\mathcal{E}$ of dimension less than or equal to $n-2$ (namely, such that $\mathcal{H}^{n-2+\varepsilon}(\Sigma(\mathcal{E})) = 0$ for every $\varepsilon > 0$) if $n \geq 3, \Sigma(\mathcal{E})$ is discrete if $n = 2$, and $\partial\mathcal{E} \setminus \Sigma(\mathcal{E})$ is a $C^{1,\alpha}$-hypersurface in $\mathbb{R}^n$ for some $\alpha \in (0, 1)$.

Let us review the theory of isoperimetric clusters when the classical perimeter, not the fractional one, is minimized. This theory has been initiated by Almgren [1] with the proof of the analogous result to Theorem 1.1, namely an existence and $C^{1,\alpha}$-regularity theorem out of a closed singular set of Hausdorff dimension $n - 1$. When $n = 2$ the only singular minimal cone consists of three half-lines meeting at 120 degrees at a common end-point, so that, by a standard dimension reduction argument, the singular set has Hausdorff dimension at most $n-2$, and is discrete when $n = 2$. (This estimate is of course sharp.) Taylor [36] has proved that, if $n = 3$, then the only singular cones are obtained either by the union of three half-planes meeting at 120 degrees along a common line, or as cones spanned by the edges of regular tetrahedra over their barycenters; and that, moreover, $\partial\mathcal{E}$ is locally $C^{1,\alpha}$-diffeomorphic to its tangent cone at every point, including singular ones. The regular part $\partial\mathcal{E} \setminus \Sigma(\mathcal{E})$ has constant mean curvature and is real analytic, in dimension $n = 3$ up to the singular set [29,25]. Regularity of and near the singular set in dimension $n \geq 4$ seems still to be mainly an open problem; in [35], Simon shows that the set of singular points where the blow up is isometric to a triple junction times $\mathbb{R}^{n-2}$, is a $n-2$-dimensional $C^{1,\alpha}$-surface.

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