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On the chain rule formulas for divergences and applications to conservation laws

Graziano Crasta[a], Virginia De Cicco[b,*]

[a] Dipartimento di Matematica "G. Castelnuovo", Univ. di Roma I, P.le A. Moro 2 – I-00185, Roma, Italy
[b] Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Univ. di Roma I, Via A. Scarpa 10 – I-00185, Roma, Italy

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To Nicola Fusco, with admiration, on the occasion of his 60th birthday

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ABSTRACT

In this paper we prove a nonautonomous chain rule formula for the distributional divergence of the composite function $\mathbf{v}(x) = \boldsymbol{B}(x, u(x))$, where $\boldsymbol{B}(\cdot, t)$ is a divergence-measure vector field and u is a function of bounded variation. As an application, we prove a uniqueness result for scalar conservation laws with discontinuous flux.

1. Introduction

Nonautonomous chain rules formulas in BV have been successfully used in the study of semicontinuity properties of integral functionals (see [12,14–16]) and conservation laws with discontinuous flux of the form

$$u_t + \operatorname{div} \boldsymbol{B}(x, u) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^N \tag{1}$$

(see [9–11] and also [20,21] in the autonomous case). In this paper we shall restrict our attention only to this second kind of application.

In order to clarify the connection between chain rule formulas and uniqueness results for the Cauchy problems associated with (1), it will be convenient to recall some previous results.

In [10] the authors considered a flux $\boldsymbol{B}$ such that $\boldsymbol{B}(\cdot, z)$ is a special function of bounded variation (SBV) and of class C^1 with respect to the second variable. A uniqueness result for (1) is then obtained in the

* Corresponding author.
E-mail addresses: crasta@mat.uniroma1.it (G. Crasta), virginia.decicco@sbai.uniroma1.it (V. De Cicco).

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class of bounded BV functions by using the chain rule formula proven in [1] for the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$. (For the sake of completeness we recall that, under the same structural hypotheses on the flux, a similar uniqueness result has been recently obtained in [11] for weak entropy solutions, without the BV regularity requirement.)

On the other hand, Panov proved in [22] an existence result of entropy solutions in the case of discontinuous fluxes $\boldsymbol{B}(x, z)$ such that $\boldsymbol{B}(\cdot, z)$ is a vector field whose distributional divergence $\operatorname{div}_x \boldsymbol{B}(\cdot, z)$ is a measure (see [6–8] for a general theory of bounded divergence-measure vector field). This assumption on $\operatorname{div}_x \boldsymbol{B}(\cdot, z)$, rather than requiring $\boldsymbol{B}(\cdot, z) \in SBV$, is indeed natural when looking for entropy solutions of (1). We remark that Panov's notion of entropy solution is somehow specific. Nevertheless, as far as we know, it is general enough to include all known notions of entropy solution.

The structure of the proof of the uniqueness result in [10] can be adapted to this more general situation, provided that one can prove a suitable chain rule formula. This is exactly the aim of this paper: in Section 4 we shall prove a nonautonomous chain rule formula for the divergence of the vector field $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$, where $\boldsymbol{B}(\cdot, t)$ is a divergence-measure vector field, of class C^1 with respect to the second variable, and $u\colon \mathbb{R}^N \to \mathbb{R}$ is a function of bounded variation. Then, we can mimic the proof in [10] in order to obtain, under these assumptions on $\boldsymbol{B}$, a uniqueness result for BV solutions of the Cauchy problems associated with (1) (see Section 5). We stress that this is not a genuine well-posedness result, since uniqueness of solutions has been proven in a class of functions which is smaller than the one for which existence has been obtained by Panov.

Before stating our results in a more precise way, let us recall the state of the art about chain rule formulas, starting from the autonomous (i.e., independent of x) case.

The first result concerning distributional derivatives is the one proved by Vol'pert in [20] (see also [21]), in view of applications to the study of quasilinear hyperbolic equations. He established a chain rule formula for distributional derivatives of the composite function $v(x) = B(u(x))$, where $u : \Omega \to \mathbb{R}$ has bounded variation in the open subset Ω of $\mathbb{R}^N$ and $B : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. He proved that v has bounded variation and its distributional derivative Dv (which is a Radon measure on Ω) admits an explicit representation in terms of the classical derivative B' and of the distributional derivative Du. More precisely, the following equality holds

$$Dv = B'(u)\nabla u\, \mathcal{L}^N + B'(\widetilde{u})D^c u + [B(u^+) - B(u^-)]\, \nu_u\, \mathcal{H}^{N-1} \llcorner J_u, \tag{2}$$

in the sense of measures, where

$$Du = \nabla u\, \mathcal{L}^N + D^c u + (u^+ - u^-)\nu_u\, \mathcal{H}^{N-1} \llcorner J_u$$

is the decomposition of Du into its absolutely continuous part $\nabla u\, \mathcal{L}^N$ with respect to the Lebesgue measure $\mathcal{L}^N$, its Cantor part $D^c u$ and its jump part, which in turn is a measure concentrated on the $\mathcal{H}^{N-1}$-rectifiable jump set J_u of u. Here, ν_u denotes the measure theoretical unit normal to J_u, $\widetilde{u}$ is the approximate limit of u and u^+, u^- are the traces of u on J_u. (Here and in the following we refer to Chapter 3 of [5] for notations and the basic facts concerning BV functions.)

An identity similar to (2) holds also in the vectorial case (see Theorem 3.96 in [5]), namely when $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^h$ has bounded variation and $B : \mathbb{R}^h \to \mathbb{R}$ is continuously differentiable. In this case, (2) can be written as

$$Dv = \nabla B(\mathbf{u})\nabla \mathbf{u}\, \mathcal{L}^N + \nabla B(\widetilde{\mathbf{u}})D^c \mathbf{u} + [B(\mathbf{u}^+) - B(\mathbf{u}^-)]\nu_{\mathbf{u}}\, \mathcal{H}^{N-1} \llcorner J_{\mathbf{u}}. \tag{3}$$

A further extension, that we are not going to use in the present paper, concerns the case when B is only a Lipschitz continuous function. In this case, a general form of the formula was proved by Ambrosio and Dal Maso in [3] (see also [18], Theorem 3.99 in [5] for the scalar case and [13] for the nonautonomous case).

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