ARTICLE IN PRESS

Nonlinear Analysis ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Regularity of minimizers under limit growth conditions

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A R T I C L E I N F O

Article history:
Received 2 June 2016
Accepted 6 June 2016
Communicated by Enzo Mitidieri

This paper is dedicated to Nicola Fusco on the occasion of his 60th birthday. Nicola is expert and master in regularity; we like here to give a small contribution to this field

MSC:
primary 49N60
secondary 35J60

Keywords:
Anisotropic growth condition
Local boundedness
Non-coercive functional

A B S T R A C T

It is well known that an integral of the Calculus of Variations satisfying anisotropic growth conditions may have unbounded minimizers if the growth exponents are too far apart. Under sharp assumptions on the exponents we prove the local boundedness of minimizers of functionals with anisotropic p, q-growth, via the De Giorgi method. As a by-product, regularity of minimizers of some non coercive functionals is obtained by reduction to coercive ones.

1. Introduction

An unusual point of view for the following integrals of the Calculus of Variations

$$\mathcal{F}(u) = \int_{B_1(0)} |x|^{\alpha} |Du|^r \, dx, \qquad \mathcal{G}(u) = \int_{B_1(0)} |x|^{-\alpha} |Du|^r \, dx, \tag{1.1}$$

with $r > 1$ and $\alpha > 0$, is to include them in the class of functionals satisfying some p, q-growth conditions. In fact, for $\mathcal{F}(u)$ in (1.1) we have that for every exponent $p \in [1, r)$,

$$|Du|^p = (|x|^{\alpha}|Du|^r)^{\frac{p}{r}} (|x|^{-\alpha})^{\frac{p}{r}} \le \frac{p}{r}|x|^{\alpha}|Du|^r + \frac{r-p}{r}|x|^{-\frac{\alpha p}{r-p}}$$

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http://dx.doi.org/10.1016/j.na.2016.06.002

and $|x|^\alpha|Du|^r \le |Du|^r$ for every $x \in B_1(0)$; so $q = r$. Hence $\mathcal{F}$, not coercive in $W^{1,r}_{\text{loc}}(B_1(0))$, is coercive in $W^{1,p}_{\text{loc}}(B_1(0))$. We claim that every local minimizer in $W^{1,p}_{\text{loc}}(B_1(0))$ of the integral $\mathcal{F}$ is locally bounded whenever

$$\begin{cases} 0<\alpha<r-1 & \text{if } 1<r\le \dfrac{n}{n-1} \\ 0<\alpha<\dfrac{r^2}{n+r} & \text{if } \dfrac{n}{n-1}<r\le n. \end{cases} \tag{1.2}$$

This result is a particular case of our Theorem 2.5, that we now state not in its full generality.

Theorem 1.1. *Let $f(x,u,\xi)$ be a Carathéodory function convex with respect to $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and such that*

$$|\xi|^p - a(x) \le f(x,u,\xi) \le L\left\{|\xi|^q + |u|^q + a(x)\right\},$$

for a.e. $x \in \Omega$, Ω open bounded set in $\mathbb{R}^n$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, for some $L > 0$ and $a \in L^s_{\text{loc}}(\Omega)$. Then, if $1 \le p \le q \le p^*$ and $s > \max\{\frac{n}{p}, 1\}$, every local minimizer of $\mathcal{F}(u) = \int_\Omega f(x,u,Du)\,dx$ in the class $W^{1,p}_{\text{loc}}(\Omega)$ is locally bounded in Ω.

Indeed, if $0 < \alpha < r - p$, the function $a(x) := |x|^{-\frac{p\alpha}{r-p}}$ is in $L^s(B_1(0))$ for some $s > \frac{n}{p}$. Since we need $r \le p^*$, if $r > \frac{n}{n-1}$ the largest upper bound on α is obtained for $p = \frac{rn}{n+r}$, so obtaining $\alpha < \frac{r^2}{n+r}$. When $r \le \frac{n}{n-1}$, the largest upper bound on α is obtained for $p = 1$.

Similarly, we can deal with the integral $\mathcal{G}$ in (1.1). In fact, for $q > r$ we have

$$|x|^{-\alpha}|Du|^r \le \frac{r}{q}|Du|^q + \frac{q-r}{q}|x|^{-\frac{\alpha q}{q-r}};$$

moreover, $|x|^{-\alpha}|Du|^r \ge |Du|^r$, for a.e. $x \in B_1(0)$. Again, by Theorem 1.1, applied with $p = r$ and $q \in (r, \frac{rn}{n-r}]$ (if $r < n$) or any $q > r$ (if $r = n$), we obtain that *every local minimizer of the integral $\mathcal{G}(u)$ in (1.1) is locally bounded* if

$$0<\alpha<\frac{r^2}{n} \quad \text{if } r \le n. \tag{1.3}$$

The functionals $\mathcal{F}$ and $\mathcal{G}$ described above are particular cases of the more general integral

$$\mathcal{F}(u) = \int_\Omega a(x)\,|Du|^r\,dx \tag{1.4}$$

with $r > 1$, $a(x) \ge 0$ a.e. in Ω, $a \in L^\sigma_{\text{loc}}(\Omega)$ and $\frac{1}{a} \in L^\tau_{\text{loc}}(\Omega)$, with $\sigma, \tau > 1$. In Theorem 6.1 we prove that, under suitable conditions on σ, τ related to n and r, see (6.2), there exist p and q, with $1 \le p \le r \le q \le p^*$, such that the integrand $f(x,Du) = a(x)\,|Du|^r$ satisfies the assumptions of Theorem 1.1 and therefore every local minimizer in $W^{1,p}_{\text{loc}}(\Omega)$ is locally bounded.

Non-uniformly elliptic equations and integrals of the Calculus of Variations of the type (1.4) with $r = 2$ have been studied by Trudinger [28] in 1971; in particular Section 3 in [28] is devoted to the study of the local boundedness of weak solutions to the Euler's equation of integrals of the type in (1.1). Higher integrability has been considered in a similar context by [5]. See also [24,25,30,8,26], and recently [21].

In this paper we consider a more general framework. In Section 2 we state our main regularity results, in particular the *local boundedness* of *minimizers* (and of *quasi-minimizers* too) of general integrals of the Calculus of Variations of the type

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u,Du)\,dx.$$

More precisely, let $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function, convex in $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$ for $|\xi|$ large enough and satisfying the following anisotropic growth condition

$$\sum_{i=1}^n [g(|\xi_i|)]^{p_i} \le f(x,u,\xi) \le L\left\{[g(|\xi|)]^q + [g(|u|)]^q + a(x)\right\}, \tag{1.5}$$