Nonlinear Analysis 158 (2017) 1–22

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Symmetry breaking bifurcations for an overdetermined boundary value problem on an exterior domain issued from electrodynamics

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ARTICLE INFO

Article history:
Received 17 March 2017
Accepted 27 March 2017
Communicated by Enzo Mitidieri

MSC:
58J32
58J05
58J55

Keywords:
Symmetry breaking
Bifurcation
Overdetermined problem
Electrically charged fluid

ABSTRACT

We consider an electrically charged fluid occupying a solid cylindrical region Ω of $\mathbb{R}^3$. Outside the domain Ω there is an electric field with electric potential which solves the Laplace equation and diverges as the distance from the axis tends to infinity. At $\partial\Omega$ the potential is constant and there is a balance between the pressure difference inside and outside the fluid, capillary forces proportional to the mean curvature and electrostatic repulsion of charges. We are interested in showing the existence of domains different from the solid cylinder Ω and satisfying the conditions described above. This problem is equivalent to an overdetermined elliptic boundary value problem on an exterior domain. We show the bifurcation phenomenon occurs and produces the deformation of the solid cylinder into rippled cylinders.

1. Introduction

In the work [4] the authors studied the existence of deformations of a spherical drop of an electrically charged fluid. Since the fluid is supposed to be conducting then the electric charge Q distributes uniformly on the surface of the drop.

If R_0 denotes the radius of the drop, H is the mean curvature of its boundary, ε_0 is the dielectric constant, then in spherical coordinates, the potential V generated by the electric charge is the solution to

$$\begin{cases} \Delta V = 0 & \text{in } \mathbb{R}^3 \setminus \Omega, \\ V = V_0 & \text{on } \partial\Omega, \\ \lim_{r\to+\infty} V(r) = 0. \end{cases} \tag{1}$$

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http://dx.doi.org/10.1016/j.na.2017.03.017

Furthermore $\delta p = \gamma H - \frac{\varepsilon_0}{2}\left(\frac{\partial V}{\partial n}\right)^2$ must hold on $\partial\Omega$, that is there is a balance between the pressure difference δp inside and outside the drop, capillary forces (proportional to the mean curvature) and electrostatic repulsion of charges (related to the normal component of the electric field, that is $\frac{\partial V}{\partial n}$). γ denotes the surface tension constant.

The authors show the existence of symmetry breaking bifurcations by which the drop becomes non-spherical for a certain limiting value of the surface tension coefficient γ. Precisely the drop becomes unstable and its shape changes when the parameter

$$X := \frac{Q^2}{32\gamma\pi^2\varepsilon_0 R_0^3},$$

is larger than a critical value X_c. Furthermore they show there exist infinitely many bifurcation branches of stationary non-spherical drops crossing, at certain values of $X = X_l$, with $l \geqslant 2$, the branch of spherical drops; here $X_c = X_2 < X_3 < \cdots$. The branch corresponding to the critical value X_2 consists in drops whose shape is a spheroid.

This is done studying an overdetermined boundary value problem on the complement of a perturbed solid ball in $\mathbb{R}^3$.

The interest in the shape of electrified drops dates back to Lord Rayleigh, who showed in [7] that if the electric charge is larger than some critical value, then a spherical drop becomes unstable, and after a finite time disintegrates into droplets of smaller size. The first experimental observation of the whole evolution process of a levitating charged drop whose radius decreases with time (by evaporation of the fluid) was described in [2]. Once the critical value of X is reached and the drop is destabilized, the drop becomes a prolate spheroid and, after a short time, develops conical tips from which two thin fluid jets are ejected. Later the jets disintegrate into smaller drops. In [1] the authors study the process of formation of conical tips for isolated charged drops. By means of a numerical calculation and asymptotic analysis, they find that droplets with Rayleigh's critical charge evolve into fusiform shapes with cones at the tips and the curvature and the velocity of the fluid at the tips diverge as $(t_0 - t)^\beta$, where β is approximately $-\frac{1}{2}$ and t_0 is the time at which the cones are formed.

In view of these discoveries it is interesting to study also charged fluid jets. For a recent survey about physics of fluid jets, see [3]. A description of the phenomena which occur when a fluid jet contains electric charges appears in Section 3.9 of [3].

It is known that cylindrical fluid jets are unstable with respect to small surface perturbations because of the development of Rayleigh instability [7] caused by capillary effects. For electrically charged fluid jets, electrostatic forces are an additional factor which determines the shape of the jet. The stability of uniformly charged fluid jets has been studied in [5]. The authors show that for the cylindrical geometry the Coulomb force has a stabilizing effect for axisymmetric sinusoidal perturbations with long wavelength and a destabilizing effect for non-axisymmetric perturbations. In [8] the authors find a wide class of equilibrium configurations of electrically charged fluid jets. These configurations correspond to the finite amplitude azimuthal deformations of the surface of the cylindrical jet.

In this work we consider a simple model: an electrically conducting fluid occupying a solid cylindrical region Ω of $\mathbb{R}^3$ of radius R_0. We assume that the fluid is electrically charged. The repulsion between the charges tends to push the surface $\partial\Omega$ outwards, therefore counteracting the cohesive action of capillarity. Since the fluid is conducting, then the electric charges are confined to the surface of the cylinder. By using the equation $\nabla \cdot E = \rho_v/\varepsilon_0$, where ρ_v denotes the charge volume density and E denotes the electric field, we get that $E = 0$ inside the cylinder because there is no electric charge in the liquid bulk. In $\mathbb{R}^3 \setminus \Omega$ there is a radial electric field

$$E = \frac{\sigma_0 R_0}{\varepsilon_0 r}\vec{e_r},$$