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A class of quasi-linear Allen–Cahn type equations with dynamic boundary conditions

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A B S T R A C T

In this paper, we consider a class of coupled systems of PDEs, denoted by $(\mathrm{ACE})_\varepsilon$ for $\varepsilon \geq 0$. For each $\varepsilon \geq 0$, the system $(\mathrm{ACE})_\varepsilon$ consists of an Allen–Cahn type equation in a bounded spacial domain Ω, and another Allen–Cahn type equation on the smooth boundary $\Gamma := \partial\Omega$, and besides, these coupled equations are transmitted via the dynamic boundary conditions. In particular, the equation in Ω is derived from the non-smooth energy proposed by Visintin in his monography "Models of phase transitions": hence, the diffusion in Ω is provided by a quasilinear form with singularity. The objective of this paper is to build a mathematical method to obtain meaningful L^2-based solutions to our systems, and to see some robustness of $(\mathrm{ACE})_\varepsilon$ with respect to $\varepsilon \geq 0$. On this basis, we will prove two Main Theorems 1 and 2, which will be concerned with the well-posedness of $(\mathrm{ACE})_\varepsilon$ for each $\varepsilon \geq 0$, and the continuous dependence of solutions to $(\mathrm{ACE})_\varepsilon$ for the variations of $\varepsilon \geq 0$, respectively.

0. Introduction

Let $0 < T < \infty, \kappa > 0$ and $N \in \mathbb{N}$ be fixed constants. Let $Q := (0,T) \times \Omega$ be a product set of a time-interval $(0,T)$ and a bounded spatial domain $\Omega \subset \mathbb{R}^N$. Let $\Gamma := \partial\Omega$ be the boundary of Ω with sufficient smoothness (when $N > 1$), and let n_Γ be the unit outer normal to Γ. Besides, we put $\Sigma := (0,T) \times \Gamma$.

In this paper, we fix a constant $\varepsilon \geq 0$ to consider the following system of PDEs, denoted by $(\mathrm{ACE})_\varepsilon$.

$(\mathrm{ACE})_\varepsilon$:

$$\partial_t u - \mathrm{div}\left(\frac{\nabla u}{|\nabla u|} + \kappa^2 \nabla u\right) + \beta(u) + g(u) \ni \theta \quad \text{in } Q, \tag{0.1}$$

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$$\partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + \left(\frac{\nabla u}{|\nabla u|} + \kappa^2 \nabla u \right)_{|_\Gamma} \cdot n_\Gamma + \beta_\Gamma(u_\Gamma) + g_\Gamma(u_\Gamma) \ni \theta_\Gamma \text{ and } u_{|_\Gamma} = u_\Gamma \quad \text{on } \Sigma, \tag{0.2}$$

$$u(0, \cdot) = u_0 \quad \text{in } \Omega, \quad \text{and} \quad u_\Gamma(0, \cdot) = u_{\Gamma,0} \quad \text{on } \Gamma. \tag{0.3}$$

The system (ACE)$_\varepsilon$ is a modified version of an Allen–Cahn type equation, proposed in [37, Chapter VI], and the principal modifications are in the points that:

- the quasi-linear (singular) diffusion in (0.1) includes the regularization term $\kappa^2 \nabla u$ with a small constant $\kappa > 0$;
- the boundary data u_Γ is governed by the dynamic boundary condition (0.2).

In general, "Allen–Cahn type equation" is a collective term to call gradient flows (systems) of governing energies, which include some double-well type potentials to reproduce the bi-stability of different phases, such as solid–liquid phases. The governing energy is called *free-energy,* and in the case of (ACE)$_\varepsilon$, the corresponding free-energy is provided as follows.

$$[u, u_\Gamma] \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) \mapsto \mathscr{F}_\varepsilon(u, u_\Gamma) := \int_\Omega \left(|\nabla u| + \frac{\kappa^2}{2} |\nabla u|^2 + B(u) + G(u) \right) dx$$
$$+ \int_\Gamma \left(\frac{\varepsilon^2}{2} |\nabla_\Gamma u_\Gamma|^2 + B_\Gamma(u_\Gamma) + G_\Gamma(u_\Gamma) \right) d\Gamma \in (-\infty, \infty], \tag{0.4}$$

with the effective domain:

$$D(\mathscr{F}_\varepsilon) := \left\{ [z, z_\Gamma] \;\middle|\; \begin{array}{l} z \in H^1(\Omega), z_\Gamma \in H^{\frac{1}{2}}(\Gamma), \varepsilon z_\Gamma \in H^1(\Gamma), \\ \text{and } z_{|_\Gamma} = z_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma) \end{array} \right\}.$$

In the context, "$|_\Gamma$" denotes the trace (boundary-value) on Γ for a Sobolev function, $d\Gamma$ denotes the area-element on Γ, ∇_Γ denotes the surface gradient on Γ, and Δ_Γ denotes the Laplacian on the surface, i.e., the so-called Laplace–Beltrami operator. $B : \mathbb{R} \to [0, \infty]$ and $B_\Gamma : \mathbb{R} \to [0, \infty]$ are given proper l.s.c. and convex functions, and $\beta = \partial B$ and $\beta_\Gamma = \partial B_\Gamma$ are the subdifferentials of B and B_Γ, respectively. $G : \mathbb{R} \to \mathbb{R}$ and $G_\Gamma : \mathbb{R} \to \mathbb{R}$ are C^1-functions, that have locally Lipschitz differentials g and g_Γ, respectively. $\theta : Q \to \mathbb{R}$ and $\theta_\Gamma : \Sigma \to \mathbb{R}$ are given heat sources of (relative) temperature, and $u_0 : \Omega \to \mathbb{R}$ and $u_{\Gamma,0} : \Gamma \to \mathbb{R}$ are initial data for the components u and u_Γ, respectively.

In (0.4), the functions:

$$\sigma \in \mathbb{R} \mapsto B(\sigma) + G(\sigma) \in (-\infty, \infty] \text{ and } \sigma \in \mathbb{R} \mapsto B_\Gamma(\sigma) + G_\Gamma(\sigma) \in (-\infty, \infty],$$

correspond to the double-well potentials, and for instance, the setting:

$$B(\sigma) = B_\Gamma(\sigma) = I_{[-1,1]}(\sigma) \quad \text{and} \quad G(\sigma) = G_\Gamma(\sigma) = -\frac{1}{2}\sigma^2, \quad \text{for } \sigma \in \mathbb{R},$$

with use of the indicator function:

$$\sigma \in \mathbb{R} \mapsto I_{[-1,1]}(\sigma) := \begin{cases} 0, & \text{if } \sigma \in [-1, 1], \\ \infty, & \text{otherwise,} \end{cases}$$

is known as one of representative choices of the components (cf. [37]).

This paper is concerned with the existence and uniqueness of the solution to (0.1)–(0.3), as well as with some continuous dependence results, also with respect to ε. Actually, our mathematical treatment of (ACE)$_\varepsilon$ is unified for the cases $\varepsilon > 0$ and $\varepsilon = 0$. Though the two cases could exhibit different features from various