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Criteria for the existence of a potential well

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ABSTRACT

We consider a critical point u_0 of a functional $f \in C^1(H, \mathbb{R})$, where H is a real Hilbert space, and formulate criteria ensuring that u_0 lies in a potential well of f without supposing that f' is Fréchet differentiable at u_0. The derivative is required to be Gâteaux differentiable at u_0, but positive definiteness of $f''(u_0)$ does not even ensure that f has a local minimum at u_0 when f' is not Fréchet differentiable at u_0. This issue is also discussed in the context of the energy functional for a parameter dependent nonlinear eigenvalue problem and then for a particular case involving a degenerate elliptic Dirichlet problem on a bounded domain in $\mathbb{R}^N$.

1. Introduction

Throughout this paper $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ is a real Hilbert space and the objective is to formulate conditions ensuring that a critical point, u_0, of a functional $f \in C^1(H, \mathbb{R})$ lies in a quadratic potential well. That is, there exist $\xi > 0$ and $r > 0$ such that

$$f(u) \geq f(u_0) + \xi \|u - u_0\|^2 \quad \text{when } \|u - u_0\| < r.$$

If $f \in C^2(H)$, this occurs when $f''(u_0)$ is positive definite. Here we deal with situations where f does have a second derivative at u_0 in the sense of Gâteaux but not necessarily in the sense of Fréchet. In cases where f' is not Fréchet differentiable at u_0, positive definiteness of the second derivative does not even ensure that u_0 is a local minimum of f. Additional conditions are formulated which imply that u_0 lies in a quadratic potential well. The existence of a potential well rather than simply a local minimum has advantages in several situations. For example, it is a crucial requirement in establishing the stability of a stationary solution of a dynamical system in infinite dimensions using a Lyapunov function. (See Section 6.6 of [10], Section 4 of [2,9] for a discussion of this issue in the context of nonlinear elasticity.) Although the existence of a potential well, or even a local minimum, is not a prerequisite for what is often referred to as the mountain pass geometry

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in critical point theory, it is nonetheless a very convenient and commonly used starting point. (See Section 8.1 of [1] and Chapter 5 of [3].)

To present the results more precisely the following more or less standard terminology is adopted. For $u_0 \in H$ and $\delta > 0$,

$$B(u_0, \delta) = \{u \in H : \|u - u_0\| < \delta\} \quad \text{and} \quad S(\delta) = \{u \in H : \|u\| = \delta\}.$$

Consider $f \in C^1(H, \mathbb{R})$ with gradient $\nabla f : H \to H$ defined by $f'(u)v = \langle \nabla f(u), v\rangle$ for all $u, v \in H$. To study the nature of a critical point of f, suppose without loss of generality that $f(0) = 0$ and $f'(0) = 0$. We have chosen to place our discussion in the context of a C^1 functional f since this is a minimal requirement for the methods used. However, it should be borne in mind that if f has one of the properties listed below and g is a functional such that $g(0) = f(0)$ and $g(u) \geq f(u)$ in some open neighbourhood of 0, then g has the same property. This places no restriction on the regularity of g.

The functional f has a local minimum at the critical point $u_0 = 0$ if there exists some $\delta > 0$ such that $f(u) \geq 0$ for all $u \in B(0, \delta)$. It is strict if $f(u) > 0$ when $0 < \|u\| < \delta$. The point 0 lies in a potential well of f if there exists $\delta > 0$ such that $m(r) > 0$ for all $r \in (0, \delta)$ where $m(r) = \inf\{f(u) : u \in S(r)\}$. The content of these definitions is clarified in a short Appendix.

A potential well is said to be quadratic if $\liminf_{r\to 0} \frac{m(r)}{r^2} > 0$. Clearly 0 lies in a quadratic potential well of f if and only if there exist $\delta > 0$ and $\xi > 0$ such that $f(u) \geq \xi\|u\|^2$ for all $u \in B(0, \delta)$. As is shown in the elementary Proposition 2.1, if $\nabla f : H \to H$ is Fréchet differentiable at 0 with a self-adjoint derivative, 0 lies in a quadratic potential well of f if and only if $f''(0)$ is positive definite. However, if ∇f is only Gâteaux differentiable, or even Hadamard differentiable, at 0, positive definiteness of $f''(0)$ does not even ensure that f has a local minimum at 0. See Section 2.1 for a simple example having this property and Corollary 4.2 for a more substantial one concerning a nonlinear Dirichlet problem. Theorems 2.2 and 2.3 give sufficient conditions for the existence of a quadratic potential well at 0 when ∇f is Hadamard differentiable at 0, without requiring Fréchet differentiability of ∇f at 0 when $\dim H = \infty$. Let us now describe the contents of this paper in little more detail.

In Section 2 we deal with the case where $\nabla f : H \to H$ is at least Gâteaux differentiable at the critical point $u_0 = 0$ with a self-adjoint derivative, T. After some elementary observations based on the Taylor expansion have been collected in Proposition 2.1, the main results of Section 2 are Theorems 2.2 and 2.3 in which ∇f is required to be Hadamard differentiable at 0. It must be acknowledged at the outset that these results can improve the conditions given in Proposition 2.1 only in cases where $\dim H = \infty$ and $\inf \sigma(T) < \inf \sigma_e(T)$, where $\sigma(T)$ and $\sigma_e(T)$ denote the spectrum and essential spectrum of T, respectively. In Theorem 2.2, f is also required to be the sum of a concave functional and a C^2- functional. No such decomposition is assumed in Theorem 2.3 but instead ∇f should be Lipschitz continuous on a neighbourhood of 0. In Section 2.1 a simple example in the space $H = L^2(0, 1)$ is considered. It shows that, in the context of these theorems, positive definiteness of $f''(0)$ does not imply that f has a local minimum at 0. Furthermore, the example shows that the additional restrictions, (2.11) in Theorem 2.2 and (2.16) in Theorem 2.3, are sharp in some cases where the more elementary criteria from Proposition 2.1 are not.

Energy functionals play an important role in the study of many nonlinear eigenvalue problems. Then the functional depends on a real parameter, λ, and the situation where $f_\lambda(0) = 0$ and $f'_\lambda(0) = 0$ for all λ is often encountered. The nature of the critical point $u_0 = 0$ will now depend upon the location of λ. Section 3 is devoted to a problem of this type in a setting which is frequently used to discuss boundary value problems for elliptic partial differential equations. An example of a problem where the energy functional is of class C^1 but the gradient is not Fréchet differentiable at the critical point is presented in Section 4. It concerns a degenerate elliptic Dirichlet problem such as

$$-\nabla \cdot \{|x|^2 \nabla u\} + V(x)u + g(x, u) = \lambda u \quad \text{for } x \in \Omega$$