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Existence of entire solutions for Schrödinger–Hardy systems involving two fractional operators

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ABSTRACT

This paper deals with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional $\wp$-Laplacian operators, via various variational methods. The main features of the paper are the presence of the Hardy terms and the fact that the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition. Moreover, we consider systems including critical nonlinear terms, as treated very recently in literature, and present radial versions of the main theorems. Finally, we briefly show how to extend the previous results when the fractional Laplacian operators are replaced by more general elliptic nonlocal integro–differential operators.

1. Introduction

The starting point of the paper is the fractional Schrödinger–Hardy system in $\mathbb{R}^n$

$$\begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu \dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v), \end{cases} \tag{1.1}$$

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where μ and σ are real parameters, $n > ps$, with $s \in (0,1), 1 < m \le p < m_s^* = mn/(n-ms)$ and $(-\Delta)_\wp^s$ is the fractional $\wp$-Laplacian operator, $\wp > 1$, which, up to normalization factors, is defined for $x \in \mathbb{R}^n$ by

$$(-\Delta)_\wp^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{\wp-2}(\varphi(x)-\varphi(y))}{|x-y|^{n+\wp s}}\, dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^n)$, see [16] and the references therein. The nonlinearities H_u and H_v denote the partial derivatives of H with respect to the second variable and the third variable, respectively, and H satisfies assumptions (H_1)–(H_4), given below.

In particular, $(-\Delta)_\wp^s$ is consistent with the fractional Laplacian $(-\Delta)^s$ as $\wp = 2$, and it is well known that $(-\Delta)_\wp^s$ reduces to the standard $\wp$-Laplacian as $s \uparrow 1$ in the limit sense of Bourgain–Brezis–Mironescu, as shown in [7].

Let us first introduce the fractional Hardy–Sobolev inequality which is basic for (1.1). Let $1 < \wp < \infty$. By Theorems 1 and 2 of [29], we know that

$$\begin{aligned} &\|u\|_{\wp_s^*}^\wp \le c_{n,\wp} \frac{s(1-s)}{(n-\wp s)^{\wp-1}} [u]_{s,\wp}^\wp, \quad \wp_s^* = \frac{\wp n}{n-\wp s}, \quad n > \wp s, \\ &\|u\|_{H_\wp}^\wp \le c_{n,\wp} \frac{s(1-s)}{(n-\wp s)^{\wp}} [u]_{s,\wp}^\wp, \qquad \|u\|_{H_\wp}^\wp = \int_{\mathbb{R}^n} |u(x)|^\wp \frac{dx}{|x|^{\wp s}}, \end{aligned} \tag{1.2}$$

for all $u \in D^{s,\wp}(\mathbb{R}^n)$, where the positive constant $c_{n,\wp}$ depends only on n and $\wp$ and $D^{s,\wp}(\mathbb{R}^n)$ is the fractional Beppo–Levi space, that is the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm $[\,\cdot\,]_{s,\wp}$ defined as

$$[\varphi]_{s,\wp} = \left(\int_{\mathbb{R}^n} |D^s \varphi(x)|^\wp \, dx \right)^{1/\wp}, \qquad |D^s\varphi(x)|^\wp = \int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^\wp}{|x-y|^{n+\wp s}} dy, \tag{1.3}$$

well defined along any test function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

A similar problem was recently studied in [35], without the Hardy terms, that is in the case $\mu = \sigma = 0$. In particular, the authors establish the existence of nontrivial nonnegative solutions of the system

$$\begin{cases} (-\Delta)_m^s u + a(x)|u|^{m-2}u = H_u(x,u,v) & \text{in } \mathbb{R}^n, \\ (-\Delta)_p^s v + b(x)|v|^{p-2}v = H_v(x,u,v) & \text{in } \mathbb{R}^n, \end{cases}$$

for which compactness arguments are easier to get than for (1.1). We recall that a nonnegative solution (u,v) is a vector function with all the components nonnegative in $\mathbb{R}^n$.

Nonlocal and fractional operators arise in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of *Lévy* processes, see for instance [3,8]. The literature on nonlocal operators and on their applications is interesting and quite large, we refer the reader to [1,9,10,13, 17,21–24,32–35] and the references therein.

In [21], the authors study a fractional problem involving a Hardy potential, subcritical and critical nonlinearities, by variational methods. The existence and regularity of a solution is provided in [1] for fractional elliptic problems with a Hardy term and different nonlinearities, even singular. By combining a variational approach and the moving plane method, in [17] the authors prove the existence and qualitative properties of a solution for a fractional problem with a critical nonlinearity and still a Hardy potential. In [24], the authors study a fractional equation in $\mathbb{R}^n$, with three critical Hardy–Sobolev nonlinearities. We refer to [10,22,23, 32] for existence results concerning different Kirchhoff–Hardy problems and Hardy–Schrödinger–Kirchhoff equations driven by the fractional Laplacian.

Regarding fractional elliptic systems, besides [35], we mention also the recent paper [13], in which the elliptic system presents only a single fractional Laplace operator and critical concave–convex nonlinearities.

Motivated by the above works, we are interested in the study of nontrivial nonnegative solutions of system (1.1) involving two fractional Laplace operators, but without the *Ambrosetti–Rabinowitz* condition. Actually,