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Local boundedness for forward–backward parabolic De Giorgi classes with coefficients depending on time

Fabio Paronetto*

Dipartimento di Matematica "Tullio Levi Civita", Università di Padova, via Trieste 63, 35121, Padova, Italy

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ABSTRACT

We define a homogeneous De Giorgi class of order $p \geqslant 2$ which suits two classes of evolution equations of the type $\mu(x,t)u_t + Au = 0$ and $(\mu(x,t)u)_t + Au = 0$, where μ can be positive, null and negative and A a suitable elliptic operator.

For functions belonging to this class we prove a local boundedness result.

1. Introduction

In this paper we consider equations whose model examples are the following

$$\frac{\partial}{\partial t}\big(\mu(x,t)u\big) - \mathrm{div}\big(|Du|^{p-2}Du\big) = 0, \qquad \mu(x,t)\frac{\partial u}{\partial t} - \mathrm{div}\big(|Du|^{p-2}Du\big) = 0, \tag{1}$$

where $p \geqslant 2$ and μ can be positive, null or negative. The aim of the paper is twofold: first we want to give a definition of a De Giorgi class that contains, besides the solutions of (1), the solution of a wide class

* Fax: +39 049 8271333.
E-mail address: fabio.paronetto@unipd.it.

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of quasi-linear mixed type equations, and then establish a result of local boundedness for the functions belonging to such a class.

An analogous class was already considered in [14] where, in addition to a Harnack type inequality, a local boundedness result was given, in the case of a function μ depending just on the x variable.

Here the interest lies in the fact that μ may depend on time and be discontinuous; for this last reason we shortly discuss the definition of the De Giorgi class.

For these types of equations the fact that μ can change sign is a complication: for instance on one hand, if μ changes its sign, it is not so useful to estimate a term like $\int u^2(x,t)\mu(x,t)\,dx$ which appears in a natural way in the energy; on the other hand it is possible to evaluate $\int u^2(x,t)\mu_+(x,t)\,dx$ only by means of $\int u^2(x,t)\mu_-(x,t)\,dx$ (μ_+ and μ_- denote the positive and the negative part of μ) and vice versa (see (53) below for details).

For this reason we consider a subclass of the "natural" De Giorgi class, for which it is possible to estimate the term $\int u^2(x,t)|\mu|(x,t)\,dx$. This class is non-empty, at least when $p=2$, by a regularity result contained in [11,15] (see Sections 3 and 4 for the details).

We consider a class which includes solutions of equations more general than those in (1) (see (29) and (30)): in particular we consider equations with some degenerate terms, also in the "elliptic part", hence we deal with weighted Sobolev spaces. We are in some sense forced to use these spaces since, also in the simplest case shown in (1), if μ takes values in $[-1,1]$, it changes sign and it is continuous, the natural setting is a weighted space. The only way to avoid weights would be to confine to $\mu : \Omega\times(0,T)\to\{-1,0,1\}$.

Here we present the class we are talking about. First define

$$\mathcal{X} := \left\{\zeta\in \mathrm{Lip}(\Omega\times(0,T))\,\middle|\; \zeta(\cdot,t)\in \mathrm{Lip}_c(\Omega) \text{ for every } t\in(0,T),\ \zeta\geqslant 0\right\} \tag{2}$$

where $\mathrm{Lip}(E)$, E open subset of $\mathbf{R}^k$, denotes the set of Lipschitz continuous functions and $\mathrm{Lip}_c(E)$ denotes the subset of $\mathrm{Lip}(E)$ of functions with compact support contained in E.

Assumptions (H.1)–(H.5) we mention in the following definition can be found in Section 4.

Definition 1.1 (*De Giorgi Class*)**.** Consider $T>0$, Ω an open subset of $\mathbf{R}^n$, $\mu,\lambda\in L^1_{\mathrm{loc}}(\Omega\times(0,T))$ satisfying (H.1)–(H.5), γ_1,γ_2 positive constants, $p\geqslant 2$. A function $u\in L^1_{\mathrm{loc}}(\Omega\times(0,T))$ such that

$$\begin{aligned}
&\iint_{\mathcal{K}}\Big[u^2|\mu| + |Du|^p\lambda\Big]dxdt < +\infty,\\
&(0,T)\ni t\mapsto \int_K |u|^2(x,t)|\mu|(x,t)\,dx \quad \text{is continuous}\\
&\text{and}\quad t\mapsto \int_K\Big[|u|^p(x,t)\lambda(x,t) + |Du|^p(x,t)\lambda(x,t)\Big]dx \quad \text{is continuous}
\end{aligned} \tag{3}$$

for every $\mathcal{K}\subset\Omega\times(0,T)$ and $K\subset\Omega$, $\mathcal{K}$ and K compact, is said to belong to the De Giorgi class $DG_+(\Omega,T,\mu,\lambda,\gamma_1,\gamma_2)$ of order p if for every function $\zeta\in\mathcal{X}$ and every $k\geqslant 0$ u satisfies

$$\begin{aligned}
&\int_\Omega (u-k)_+^2(x,t_2)\zeta^p(x,t_2)\mu(x,t_2)dx + \gamma_1\int_{t_1}^{t_2}\int_\Omega |D(u-k)_+|^p\zeta^p\lambda\,dxdt\\
&\leqslant \int_\Omega (u-k)_+^2(x,t_1)\zeta^p(x,t_1)\mu(x,t_1)dx + \gamma_2\Bigg[\int_{t_1}^{t_2}\int_\Omega\Big[(u-k)_+^p|D\zeta|^p\lambda + (u-k)_+^2\zeta^{p-1}\zeta_t\,\mu\Big]dxdt\\
&\quad + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^p\zeta^p\lambda\,dxdt + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^2\zeta^p\lambda\,dxdt\\
&\quad + k^p\int_{t_1}^{t_2}\int_{\{u(t)>k\}}\zeta^p\lambda\,dxdt + \delta_2(p)\,k^2\int_{t_1}^{t_2}\int_{\{u(t)>k\}}|D\zeta|^2\lambda\,dxdt\Bigg]
\end{aligned} \tag{4}$$