Nonlinear Analysis 157 (2017) 44–61

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Norm inflation for the generalized Boussinesq and Kawahara equations

Mamoru Okamoto

Division of Mathematics and Physics, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano City 380-8553, Japan

A R T I C L E I N F O

Article history:
Received 27 July 2016
Accepted 20 March 2017
Communicated by Enzo Mitidieri

MSC:
35Q55
35B30

Keywords:
Boussinesq equation
Kawahara equation
Ill-posedness
Norm inflation
General initial data

A B S T R A C T

We consider ill-posedness of the Cauchy problem for the generalized Boussinesq and Kawahara equations. We prove norm inflation with general initial data, an improvement over the ill-posedness results by Geba et al. (2014) for the generalized Boussinesq equations and by Kato (2011) for the Kawahara equation.

1. Introduction

We consider the Cauchy problem for the generalized Boussinesq equation

$$\begin{aligned} &\partial_t^2 u - \Delta u + \Delta^2 u + \Delta(N(u)) = 0, \\ &u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x) \end{aligned} \tag{1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an unknown function, and u_0 and u_1 are given functions. Falk et al. [7] derived this equation for $d = 1$ with $N(u) = 4u^3 - 6u^5$ in a study of shape-memory alloys. For $N(u) = u^2$, this is the "good" Boussinesq equation, which arises as a model for nonlinear strings [27].

In the sequel, we consider (1) with $N(u) = u^p$. If we ignore Δu, (1) is invariant under the scaling transformation $u \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$. From

$$\|u_\lambda(0,\cdot)\|_{\dot{H}^s} = \lambda^{s-\frac{d}{2}+\frac{2}{p-1}} \|u_0\|_{\dot{H}^s},$$

E-mail address: m_okamoto@shinshu-u.ac.jp.

http://dx.doi.org/10.1016/j.na.2017.03.011

we call the index $s_c := \frac{d}{2} - \frac{2}{p-1}$ scaling-critical, although the generalized Boussinesq equation does not have the exact scaling invariance.

Well-posedness of (1) has been studied intensively for $d = 1$ (see [2,19,8,9,18,17] and references therein). Farah [8] proved that (1) with $d = 1$ and $N(u) = u^p$ is well-posed in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ if $d = 1$, $p > 1$, and $s \geq \max(s_c, 0)$. Kishimoto [17] showed that (1) with $d = 1$ and $N(u) = u^2$ is well-posed in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ if $s \geq -\frac{1}{2}$. He also proved that this result is sharp in the sense that the flow map $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$ of (1) fails to be continuous at zero if $s < -\frac{1}{2}$.

Geba et al. [10] proved that the flow map $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$ of (1) fails to be p-times differentiable at zero if

$$s < \begin{cases} -\dfrac{2}{p}, & \text{for } p \text{ odd}, \\ -\dfrac{1}{p}, & \text{for } p \text{ even}. \end{cases}$$

It is known that the flow map is smooth if we obtain well-posedness through an iteration argument [1]. Hence, they showed that the standard iteration argument fails to work for (1). However, as well-posedness involves the continuity of the flow map, there is a gap between ill-posedness and the presence of an irregular flow map. In this paper, we prove ill-posedness of (1) by observing norm inflation.

Theorem 1.1. *Let $d \in \mathbb{N}$, $p \in \mathbb{Z}_{\geq 2}$, and $N(u) = u^p$. Assume that one of the following holds:*

- $d = 1$, $p = 3$, $s \leq -\frac{1}{2}$.
- $d \in \mathbb{N}$, $p = 2$, $s < -\frac{1}{2}$.
- $d \in \mathbb{N}$, $p \geq 3$, $s < \min(s_c, 0)$.

For any $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-2}(\mathbb{R}^d)$, and any $\varepsilon > 0$, there exists a solution u_ε to (1) and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u_\varepsilon(0) - u_0\|_{H^s} + \|\partial_t u_\varepsilon(0) - u_1\|_{H^{s-2}} < \varepsilon, \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}.$$

In particular, the flow map of (1) is discontinuous everywhere in $H^s(\mathbb{R}^d) \times H^{s-2}(\mathbb{R}^d)$.

Theorem 1.1 is an improvement of the result by Geba et al. in terms of the property of the flow map and the range of s.

We set $v := u - i(1 - \Delta)^{-1}\partial_t u$. Since u is real valued, (1) is equivalent to

$$\begin{aligned} &i\partial_t v - \Delta v = -\frac{1}{2}(v - \overline{v}) + \frac{1}{2^p}\omega(\sqrt{-\Delta})(v + \overline{v})^p, \\ &v(0, x) = v_0(x), \end{aligned} \tag{2}$$

where $\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$ and $v_0 = u_0 - i(1-\Delta)^{-1}u_1$. The restriction to real-valued functions is not essential, but assumed here for simplicity. When u is complex-valued, (1) is reduced to a system of nonlinear Schrödinger equations, and the same ill-posedness result holds (see Remark 3.1).

Since $\omega(-\sqrt{\Delta})$ is bounded in $L^2(\mathbb{R}^d)$, we can neglect it and reduce (2) to the Schrödinger equation with the power type nonlinearity. Hence, the same calculation as the nonlinear Schrödinger equation yields well-posedness of (1). In contrast, from $\omega(\xi) \sim |\xi|$ for $|\xi| < 1$, (1) with $d = 1$ and $p = 2$ is well-posed in $H^{-\frac{1}{2}}(\mathbb{R})$, although Kishimoto and Tsugawa [18] proved that well-posedness in $H^s(\mathbb{R})$ for the nonlinear Schrödinger equation with $|u|^2$ holds if and only if $s \geq -\frac{1}{4}$.

Iwabuchi and Ogawa [12] developed a method for proving ill-posedness of evolution equations using the modulation space. This method is a refinement of previous work by Bejenaru and Tao [1]. Recently, many