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Hardy–Sobolev spaces associated with Hermite expansions and interpolation☆

Jizheng Huang

College of Sciences, North China University of Technology, Beijing, 100144, China

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ABSTRACT

Let $A = -\Delta + |x|^2$ be a Hermite operator, where Δ is the Laplacian on $\mathbb{R}^d$. In this paper, we first define the Hardy–Sobolev spaces associated with Hermite expansions, then we give the atomic decomposition and the real interpolation with Sobolev spaces. As an application, we consider the endpoint version of the div–curl theorem for the Hermite operator.

1. Introduction

It is a well-established fact that, for the purposes of harmonic analysis or theory of partial differential equations, the right substitute for $L^p(\mathbb{R}^d)$ in case $p \in (0, 1]$ is the (real) Hardy space $H^p(\mathbb{R}^d)$, or its local version $h^p(\mathbb{R}^d)$ (cf. [17]). The Hardy spaces, or their local versions if needed, behave nicely under the action of regular singular integrals or pseudo-differential operators. Moreover, in the case of Hardy spaces the Littlewood–Paley theory and interpolation results extend to the whole scale of Lebesgue exponents $p \in (0, \infty)$. It is hence natural to investigate Sobolev spaces where one (roughly speaking) demands that the sth derivative belongs to a Hardy type space in the case $p \leq 1$. After the fundamental work of Fefferman and Stein [15] this line of research was initiated by Peetre in early 70s, and it was generalized and carried further by Triebel and others. We refer to [24,32] for extensive accounts on general Besov and Triebel-type scales of function spaces in the case $p \in (0, 1]$.

Let $I_\alpha : \mathcal{S}'/\mathcal{P} \to \mathcal{S}'/\mathcal{P}$ be the Riesz potential operators, where $\mathcal{S}'$ is the space of tempered distributions and $\mathcal{P}$ denotes the space of polynomials. We can define Sobolev space $I_\alpha(L^p)(p > 1)$ to be the space of

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E-mail address: hjzheng@163.com.

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tempered distributions having derivatives of order α in L^p. The use of the Hardy–Sobolev spaces gives strong boundedness of some linear operators instead of the weak boundedness. For instance this is the case of the square root of the Laplace operator $\triangle^{1/2}$. The Hardy–Sobolev spaces were studied by many authors. In [28], the author investigated the spaces $I_\alpha(H^p)(0 < p \leq 1)$, where H^p denotes the Hardy spaces. The spaces H^p form a natural continuation of the L^p space to $0 < p \leq 1$, and so the spaces $I_\alpha(H^p)$ which are called Hardy–Sobolev spaces are natural generalizations of the homogeneous Sobolev spaces $I_\alpha(L^p)$ to the range $0 < p \leq 1$. Strichartz [28] proved that $I_{n/p}(H^p)$ was an algebra and found equivalent norms for the Hardy–Sobolev spaces or, more generally, for corresponding spaces with fractional smoothness and Lebesgue exponents in the range $p > n/(n+1)$. Torchinsky [31] discussed the trace properties of the spaces $I_\alpha(H^p)$. Miyachi [22] characterized the Hardy–Sobolev spaces in terms of maximal functions related to mean oscillation of the function in cubes, thus obtaining a counterpart of previous results of Calderon and of the general theory of DeVore and Sharpley [11]. More recently there has been considerable interest in Hardy–Sobolev spaces and their variants on $\mathbb{R}^d$, or on subdomains. Chang, Dafni, and Stein [8] consider Hardy–Sobolev spaces in connection with estimates for elliptic operators, whereas Auscher, Emmanuel, and Tchamitchian [1] study these spaces with applications to square roots of elliptic operators. Koskela, Saksman [20] show that there is a simple strictly pointwise characterization of the Hardy–Sobolev spaces in terms of first differences. In [21], the authors gave the atomic decomposition of the Hardy–Sobolev space and proved the endpoint case of the div–curl theorem of [10]. Also the papers of Gatto, Segovia, and Jimenez [16], Cho, Kim [9], Jason [19] and Orobitg [23] are related to the theme of the present paper. In [4,5], the authors consider the Hardy–Sobolev spaces on the manifold.

Recently, functional spaces associated with operators are considered by more and more mathematicians. In [7], the authors studied the Sobolev spaces associated with the Hermite operator. Then they considered the boundedness of operators and almost everywhere convergence of solutions of the Schrödinger equation associated with Hermite operator. In [13], the authors defined the Hardy spaces associated with Hermite operator by the heat maximal function. They also gave the atomic decomposition and Riesz transform characterizations for the Hardy spaces. In this paper, we first define Hardy–Sobolev spaces associated with Hermite operator based on [7,13], then give the atomic decomposition of them. Finally, we consider real interpolation of them with Sobolev spaces.

The paper is organized as follows. In Section 2, we give some results that we will use in the sequel; In Section 3, we prove some properties of the Hardy–Sobolev space, including atomic decomposition. In Section 4, some applications will be given. In Section 5, the real interpolation of Hardy–Sobolev spaces and Sobolev spaces will be considered.

2. Preliminaries

The Hermite polynomial on the real line is defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k = 0, 1, 2, \ldots.$$

Then, the Hermite function is defined by

$$h_k(x) = \left(\pi^{1/2} 2^k k!\right)^{-1/2} H_k(x) \exp\left(-x^2/2\right), \quad k = 0, 1, \ldots.$$

In the d-dimensional Euclidean space $\mathbb{R}^d$, the Hermite functions are defined as follows. For any multi-index α and $x \in \mathbb{R}^d$, we define

$$h_\alpha(x) = \prod_{j=1}^{d} h_{\alpha_j}(x_j),$$