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Elliptic gradient estimates for a nonlinear heat equation and applications

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ABSTRACT

In this paper, we study elliptic gradient estimates for a nonlinear f-heat equation, which is related to the gradient Ricci soliton and the weighted log-Sobolev constant of smooth metric measure spaces. Precisely, we obtain Hamilton's and Souplet–Zhang's gradient estimates for positive solutions to the nonlinear f-heat equation only assuming the ∞-Bakry–Émery Ricci tensor is bounded below. As applications, we prove parabolic Liouville properties for some kind of ancient solutions to the nonlinear f-heat equation. Some special cases are also discussed.

1. Introduction

1.1. Background

In the previous work [29], we proved elliptic gradient estimates for positive solutions to the f-heat equation on smooth metric measure spaces with only the ∞-Bakry–Émery Ricci tensor bounded below. We also applied the results to get parabolic Liouville theorems for some ancient solutions to the f-heat equation. In this paper we will investigate elliptic gradient estimates and Liouville properties for positive solutions to a nonlinear f-heat equation (see Eq. (1.3)) on complete smooth metric measure spaces.

Recall an n-dimensional smooth metric measure space $(M^n, g, e^{-f}dv)$ is a complete Riemannian manifold (M^n, g) endowed with a weighted measure $e^{-f}dv$ for some $f \in C^\infty(M)$, where dv is the volume element of

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the metric g. The associated m-Bakry–Émery Ricci tensor [2] is defined by

$$Ric_f^m := Ric + \nabla^2 f - \frac{1}{m} df \otimes df$$

for some constant $m > 0$, where Ric and ∇^2 denote the Ricci tensor and the Hessian of the metric g. When $m = \infty$, we have the (∞-)Bakry–Émery Ricci tensor

$$Ric_f := Ric + \nabla^2 f.$$

The Bochner formula for Ric_f^m can be read as (see also [29])

$$\begin{aligned}\frac{1}{2}\Delta_f |\nabla u|^2 &= |\nabla^2 u|^2 + \langle \nabla \Delta_f u, \nabla u\rangle + Ric_f(\nabla u, \nabla u)\\ &\geq \frac{(\Delta_f u)^2}{m+n} + \langle \nabla \Delta_f u, \nabla u\rangle + Ric_f^m(\nabla u, \nabla u)\end{aligned} \tag{1.1}$$

for any $u \in C^\infty(M)$. When $m < \infty$, (1.1) could be viewed as the Bochner formula for the Ricci tensor of an $(n+m)$-dimensional manifold. Hence many geometric and topological properties for manifolds with Ricci tensor bounded below can be possibly extended to smooth metric measure spaces with m-Bakry–Émery Ricci tensor bounded below, see for example [17,22]. When $m = \infty$, the (∞-)Bakry–Émery Ricci tensor is related to the gradient Ricci soliton

$$Ric_f = \lambda\, g$$

for some constant λ, which plays an important role in Hamilton's Ricci flow as it corresponds to the self-similar solution and arises as limits of dilations of singularities in the Ricci flow [12]. A Ricci soliton is said to be shrinking, steady, or expanding according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. On the gradient Ricci soliton, the smooth function f is often called a potential function. We refer [5] and the references therein for further discussions.

On smooth metric measure space $(M, g, e^{-f}dv)$, the f-Laplacian Δ_f is defined by

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to the weighted measure. The associated f-heat equation is defined by

$$\frac{\partial u}{\partial t} = \Delta_f\, u. \tag{1.2}$$

If u is independent of time t, then it is f-harmonic function. In the past few years, various Liouville properties for f-harmonic functions were obtained, see for example [3,17,19,24,26,28,30,31], and the references therein. Recently, the author [29] proved elliptic gradient estimates and parabolic Liouville properties for f-heat equation under some assumptions of (∞-)Bakry–Émery Ricci tensor.

In this paper, we will study analytical and geometrical properties for positive solutions to the general equation

$$\frac{\partial u}{\partial t} = \Delta_f\, u + au \ln u, \tag{1.3}$$

where $a \in \mathbb{R}$, on complete smooth metric measure spaces $(M, g, e^{-f}dv)$ with only the Bakry–Émery Ricci tensor bounded below. Here we assume M has no boundary. It is well-known that all solutions to its Cauchy problem exist for all time. Under the assumption of Ric_f, we shall prove local elliptic (Hamilton's type and Souplet–Zhang's type) gradient estimates for positive solutions to the nonlinear f-heat equation (1.3). As applications, we prove parabolic Liouville properties for the nonlinear f-heat equation (1.3).

Historically, gradient estimates for the harmonic function on manifolds were discovered by Yau [33] and Cheng-Yau [7] in 1970s. It was extended to the so-called Li–Yau gradient estimate for the heat equation by