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A priori bounds for superlinear elliptic equations with semidefinite nonlinearity

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ABSTRACT

We derive a priori bounds for positive solutions of the superlinear elliptic problems $-\Delta u = a(x)u^p$ on a bounded domain Ω in $\mathbf{R}^N$, where $a(x)$ is Hölder continuous in Ω. Our main motivation is to study the case where $a(x) \geq 0$, $a(x) \not\equiv 0$ and $a(x)$ has some zero sets in Ω. We show that, in this case, the scaling arguments reduce the problem of a priori bounds to the Liouville-type results for the equation $-\Delta u = A(x')u^p$ in $\mathbf{R}^N$, where A is the continuous function defined on the subspace $\mathbf{R}^k$ with $1 \leq k \leq N$ and $x' \in \mathbf{R}^k$. We also establish a priori bounds of global nonnegative solutions to the corresponding parabolic initial–boundary value problems.

1. Introduction

We consider a priori bounds of positive solutions to the elliptic problems

$$\begin{cases} -\Delta u = a(x)u^p & \text{in } \Omega, \\ u(x) = \phi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $p > 1$, $a(x)$ is Hölder continuous and satisfies $a(x) \geq 0$, $a(x) \not\equiv 0$ in $\overline{\Omega}$, and $\phi \in C(\partial\Omega)$ satisfies $\phi \geq 0$ on $\partial\Omega$.

In order to prove existence and multiplicity of positive solutions to (1.1), it is important to obtain a priori bounds for its positive solutions (see, e.g. [4]). It is well known that Liouville-type results enable one

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to derive a priori bounds for positive solutions of elliptic problems via the rescaling method. In the case where $a(x)$ is continuous and strictly positive on $\overline{\Omega}$, a priori bounds was obtained by Gidas and Spruck [7]. The method in [7] is based on rescaling and Liouville theorems for nonnegative solutions to the equation $-\Delta u = u^p$ either in $\mathbf{R}^N$ or in a half-space with 0 boundary condition.

Our main goal here is to establish similar results as in [7] for the semidefinite function $a(x)$, namely, $a(x) \geq 0$, $a(x) \not\equiv 0$ in Ω and $a(x)$ has some zero sets in Ω. In [7], the uniform positivity of $a(x)$ plays a crucial role to derive the entire space problem $-\Delta u = u^p$ in $\mathbf{R}^N$ or in a half-space. We will show that, in the semidefinite case, the scaling arguments reduce the problem of a priori bounds to the Liouville-type results for nonnegative solutions to the equation

$$-\Delta u = A(x')u^p \quad \text{in } \mathbf{R}^N,$$

where A is the continuous function defined on $\mathbf{R}^k$ with $1 \leq k \leq N$ and $x' \in \mathbf{R}^k$.

Define $Z = \{x \in \overline{\Omega} : a(x) = 0\}$. First we consider the case $Z = \overline{\Omega} \setminus \omega$, where $\omega \subset \mathbf{R}^N$ is an open set with C^1-boundary $\partial\omega$ satisfying $\overline{\omega} \subset \Omega$. For $x_0 \in \partial\omega$, $\nabla a(x_0)$ stands for $\lim_{x\in\omega, x\to x_0} \nabla a(x)$. We denote by $\xi(x_0)$ the outer unit normal vector on $\partial\omega$ at x_0. We consider the following case:

(A) In (1.1), $a(x)$ is Hölder continuous and satisfies $a(x) \geq 0$, $a(x) \not\equiv 0$ in $\overline{\Omega}$. The set Z is given by $Z = \overline{\Omega} \setminus \omega$, where $\omega \subset \mathbf{R}^N$ is an open set with C^1 smooth boundary $\partial\omega$ satisfying $\overline{\omega} \subset \Omega$. The following either (i) or (ii) must hold.

(i) $a \in C^1(\overline{\omega})$ and $\nabla a(x) \neq 0$ for all $x \in \partial\omega$.
(ii) $a \in C^2(\overline{\omega})$ and $\nabla a(x) = 0$ for all $x \in \partial\omega$. For each $x_0 \in \partial\omega$, one has

$$\lim_{s\to 0+} \frac{d^2}{ds^2} a(x_0 - s\xi(x_0)) \neq 0. \tag{1.2}$$

Remark 1.1. Assume that (A) holds. Since $a(x) \geq 0$ in Ω, if $a \in C^1(\Omega)$, then $\nabla a(x) = 0$ for all $x \in \partial\omega$. The condition (1.2) requires that the Hessian matrix of $a(x)$ is not 0 at $x = x_0$. In fact, we see that

$$\lim_{s\to 0+} \frac{d^2}{ds^2} a(x_0 - s\xi(x_0)) = \sum_{|\alpha|=2} \partial_x^\alpha a(x_0)(-\xi(x_0))^\alpha,$$

where α is the multi-index (see Section 3). If $a(x)$ is given by a nonnegative part of some function, $a(x)$ may satisfy (i) of (A).

Theorem 1.1. *Assume that $1 < p < (N+2)/(N-2)$ and (A) holds. Let $M > 0$, and let ϕ satisfy $\|\phi\|_{L^\infty(\partial\Omega)} \leq M$. Then any positive solutions u of (1.1) satisfy $\|u\|_{L^\infty(\Omega)} \leq C$, where the constant $C > 0$ depends only on Ω, a, p and M.*

Remark 1.2. (i) In Theorem 1.1, we can replace the condition (A) by that a satisfies, for each $x_0 \in \partial\omega$,

$$a(x) = \chi_{\overline{\omega}}(x) \cdot \left(\sum_{|\alpha|=m_0} \frac{1}{m_0!} \partial_x^\alpha a(x_0)(x - x_0)^\alpha + o(|x - x_0|^{m_0}) \right) \quad \text{as } x \to x_0$$

with

$$\sum_{|\alpha|=m_0} \partial_x^\alpha a(x_0)(-\xi(x_0))^\alpha \neq 0,$$

where χ_A is the characteristic function of $A \subset \mathbf{R}^N$ and $m_0 = 1$ or 2. (See Section 3.)