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Mixed Morrey spaces and their applications to partial differential equations

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ABSTRACT

In this paper, new classes of functions are defined. These spaces generalize Morrey spaces and give a refinement of Lebesgue spaces. Some embeddings between these new classes are also proved. Finally, the authors apply these classes of functions to obtain regularity results for solutions of partial differential equations of parabolic type.

1. Introduction

This paper aims at defining new spaces and to study some embeddings between them. We will refer to them with the symbol $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. As applications we obtain some estimates, in these classes of functions, for the solutions of partial differential equations of parabolic type in nondivergence form. Preparatory to achieving these results is the study of the behavior of Hardy–Littlewood Maximal function, Riesz potential, Sharp and Fractional maximal functions, Singular integral operators with Calderón–Zygmund kernel and Commutators (see e.g. [22,23]).

We stress that are obtained results, known in L^p, in a new class of functions that can be view as an extension of the Morrey class introduced in 1966 in [17], and used by a lot of authors, see e.g. in [3], recently in [24,20,12,14,13] and others.

Let us point out that in doing this we need an extension to $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ of a celebrated inequality of Fefferman and Stein (see [10]) concerning the Sharp and the Maximal function (Theorem 4.5) and, also, we study the behavior of Riesz potential in the new class of functions, obtaining an extension of both a known estimate originally proved by Adams in [1] as well as of a result announced by Peetre in [19].

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2. Definitions and preliminary tools

In the sequel let $T > 0$ and Ω be a bounded open set of $\mathbb{R}^n$ such that $\exists A > 0 : \forall x \in \Omega$ and $0 \le \rho \le \; diam\,(\Omega)$, $|Q(x,\rho) \cap \Omega| \ge A\,\rho^n$, being $Q(x,\rho)$ a cube centered in x, having edges parallel to the coordinate axes and length 2ρ.

Definition 2.1. Let $1 < p < +\infty,\; 0 < \lambda < n$ and f be a real measurable function defined in $\Omega \subset \mathbb{R}^n$.

If $|f|^p$ is summable in Ω and the set described by the quantity

$$\frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y)|^p \, dy, \tag{2.1}$$

when changing of ρ in $]0,\; diam\,\Omega[$ and $x \in \Omega$, has an upper bound, then we say that f belongs to the *Morrey Space* $L^{p,\lambda}(\Omega)$.

If $f \in L^{p,\lambda}(\Omega)$, we define

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y)|^p \, dy \tag{2.2}$$

and the vector space naturally associated to the set of functions in $L^p(\Omega)$ such that (2.2) is finite, endowed with the norm (2.2), is a normed and complete space.

The exponent λ can take values that are not belonging to $]0,n[$ but the unique cases of real interest are that one for which $\lambda \in]0,n[$.

The above defined spaces are used, among others, in the theory of regular solutions to nonlinear partial differential equations and for the study of local behavior of solutions to nonlinear equations and systems (see e.g. [17,18]).

Remark 2.1. Similarly we can define the Morrey space in $L^{p,\lambda}(\mathbb{R}^n)$ as the space of functions such that is finite:

$$\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y)|^p \, dy. \tag{2.3}$$

Definition 2.2. Let $1 < p, q < +\infty,\;\; 0 < \lambda, \mu < n$. We define the set $L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))$ as the class of functions f such that is finite:

$$\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} := \left(\sup_{\substack{t_0, t\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{\Omega\cap B_\rho(x)} |f(y,t)|^p \, dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}, \tag{2.4}$$

with obvious modifications if $\Omega = \mathbb{R}^n$.

Definition 2.3. Let f be a locally integrable function defined on $\mathbb{R}^n$. We say that f is in the space $BMO(\mathbb{R}^n)$ (see [15]) if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |f(y) - f_B| dy < \infty$$

where B runs over the class of all balls in $\mathbb{R}^n$ and $f_B = \frac{1}{|B|} \int_B f(y)dy$.