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Infinitely many solutions for sublinear indefinite nonlocal elliptic equations perturbed from symmetry[☆]

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ABSTRACT

This paper is concerned the existence of infinitely many solutions for perturbed sublinear indefinite elliptic equations involving the nonlocal operator. By using a variant of Rabinowitz's perturbation method, we study the effect of high order non-odd perturbations which break the symmetry of the associated energy functional and show how a sequence of small negative energy solutions persists under broken symmetry situations.

1. Introduction and main results

Consider the following problem

$$\begin{cases} -\mathcal{L}_K u = g(x,u) + h(x,u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$). Moreover, $\mathcal{L}_K$ is the nonlocal operator defined as follows

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)dy, \quad x \in \mathbb{R}^N,$$

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where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function possessing the following properties:

$(K1)$ $mK \in L^1(\mathbb{R}^N)$, where $m(x) := \min\{|x|^2, 1\}$;
$(K2)$ there exist $\rho > 0$ and $s \in (0, 1)$ such that $K(x) \geq \rho|x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$;
$(K3)$ $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

A typical example for the kernel function K is given by $K(x) = \rho|x|^{-(N+2s)}$. In this case, $\mathcal{L}_K$ is the fractional Laplacian operator $-(-\triangle)^s$, which (up to normalization factors) is defined as

$$-(-\triangle)^s = \frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

The interest for nonlocal operators and their applications to differential equations has increased in recent years. These nonlocal operators such as $(-\triangle)^s$ arise naturally in quantum mechanics, anomalous diffusion, phase transitions, crystal dislocation and game theory, as they can be seen as the infinitesimal generator of Lévy stable diffusion processes, see [1,5,16,18,19,33] and the references therein.

Next we introduce some functional spaces in order to state our main results. Let X denote the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x, y) \mapsto (u(x) - u(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big)$$

where $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$. Moreover, $X_0 := \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$. By Lemma 5.1 in [30], $C_0^2(\Omega) \subseteq X_0$, so X and X_0 are non-empty.

We say that $u \in X_0$ is a weak solution of problem (1.1), if u satisfies

$$\int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)dxdy = \int_\Omega \Big(g(x, u(x)) + h(x, u(x))\Big)\varphi(x)dx, \quad \forall\, \varphi \in X_0.$$

Since problem (1.1) has a variational nature and its weak solutions can be constructed as critical points of the associated energy functional, much attention has been focused on the existence and multiplicity of weak solutions for problem (1.1) and its variant by variational methods (see, e.g., [8,20,29,31,32,36,38]). Very recently, Fiscella and Valdinoci [13] first proposed a stationary Kirchhoff variational model involving nonlocal integro-differential operators. By the use of assigned Krasnoselskii's genus on a symmetric compact set, Figueiredo, Bisci and Servadei [11] showed the existence of infinitely many solutions for Kirchhoff type problems involving the nonlocal operator $\mathcal{L}_K$ and a nonlocal subcritical term. Besides, in combination with the classical Krasnoselskii's genus theory and truncation argument, Fiscella [12] exploited a concentration compactness principle to overcome the lack of compactness at critical level and proved a sequence of solutions for a critical fractional Kirchhoff type problem. For some more multiplicity results in this direction, for example, see [9,22,37] and the references therein. In these works, the methods rely on the use of Lusternik–Schnirelmann theory or rather on the notion of genus for symmetric sets. Therefore, the fact that the symmetry of the corresponding functional plays an essential role in the application of these techniques. A natural problem is whether the infinite number of solutions persists in broken symmetry case, and this problem is often referred to as the perturbation from symmetry problem.

Since the early 1980s, some mathematicians have developed different methods to investigate the problem of perturbation from symmetry. In particular we mention Bahri and Berestycki [2], Struwe [34], Rabinowitz [23], Bahri and Lions [3], Bolle [4]. Roughly speaking, they proved that the existence of infinitely many solutions can be preserved by restricting the growth range of non-odd perturbations with suitable bounds. The study concerning the perturbation problem for classical elliptic equations and systems has attracted great interest in the last years, and there has been much work on this topic (see, e.g., [6,15,17,25–27,35,40–42]). However, few results are known for elliptic equations involving nonlocal operators for the perturbation problem. As