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A semilinear PDE with free boundary

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ABSTRACT

We study the semilinear problem

$$\Delta u = \lambda_+(x)(u^+)^{q-1} - \lambda_-(x)(u^-)^{q-1} \qquad \text{in } B_1,$$

from a regularity point of view for solutions and the free boundary $\partial\{\pm u > 0\}$. Here B_1 is the unit ball, $1 < q < 2$ and $\lambda_\pm$ are Lipschitz.

Our main results concern local regularity analysis of solutions and their free boundaries. One of the main difficulties encountered in studying this equation is classification of global solutions. In dimension two we are able to present a fairly good analysis of global homogeneous solutions, and hence a better understanding of the behavior of the free boundary. In higher dimensions the problem becomes quite complicated, but we are still able to state partial results; e.g. we prove that if a solution is close to one-dimensional solution in a small ball, then in an even smaller ball the free boundary can be represented locally as two C^1-regular graphs $\Gamma^+ = \partial\{u > 0\}$ and $\Gamma^- = \partial\{u < 0\}$, tangential to each other.

It is noteworthy that the above problem (in contrast to the case $q = 1$) introduces interesting and quite challenging features, that are not encountered in the case $q = 1$. E.g. one obtains homogeneous global solutions that are not one-dimensional. This complicates the analysis of the free boundary substantially.

1. Introduction

1.1. Problem setting

Let B_1 be the unit ball in $\mathbb{R}^n$ $(n \geq 2)$, $g \in W^{1,2}(B_1) \cap L^\infty(B_1)$, and consider the minimizer(s) of the functional

$$J(u) = \int_{B_1} \left(|\nabla u|^2 + 2F(x,u)\right) dx, \quad u - g \in W_0^{1,2}(B_1), \tag{1}$$

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where

$$F(x,u) = \frac{1}{q}\big(\lambda_+(x)(u^+)^q + \lambda_-(x)(u^-)^q\big),$$

with $u^\pm = \max\{\pm u, 0\}$. Throughout the paper we shall use standing conditions on $\lambda_\pm$, and q as follows:

H1. There is a λ_0 such that $\lambda_\pm(x) \geq \lambda_0 > 0$.
H2. $q \in (1,2)$.
H3. $\lambda_\pm(x) \in C^{0,1}$.

We should remark that the above problem with $1 < q < 2$, is qualitatively of different character than related problems when $q > 2$, see [15].

It is straightforward that (1) has a unique (bounded) minimizer which satisfies the following semilinear equation in the weak sense

$$\Delta u = f(x,u) := \lambda_+(x)(u^+)^{q-1} - \lambda_-(x)(u^-)^{q-1}, \quad \text{in } B_1. \tag{2}$$

A solution to this equation (regardless of the boundary data) belongs to $C^{1,\alpha}(B_{1/2})$, and therefore due to the structure of the right hand side $f(x,u)$ in (2) we will have $f(x,u)$ is Hölder of order $(q-1)$. Hence by elliptic theory, solutions are $C^{2,(q-1)}$. On the other hand if at a point $z \in \Gamma = \{u = 0\}$, we also have that the derivatives of u vanish at z, then one can raise the regularity at such a point to $C^{2,(2(q-1))}$, provided $\lambda_\pm \in C^{0,2(q-1)}$; see e.g. [7, Theorem 8.1]. Nevertheless, the solution will continue to have slightly lower regularity $C^{2,(q-1)}$ at other free boundary points where some of the derivatives do not vanish. This feature of the solutions complicates the matter and the local analysis of the free boundary becomes much more delicate. In particular, the above two-phase problem does not follow the same lines of behavior as that of one-phase case, where one assumes the solutions to be non-negative; see [3,11,14,13].[1]

Remark 1.1. It should be remarked that close to those points where u does not decay of highest order (which in our problem is $\kappa := 2/(2-q)$) the solution behaves like a harmonic function. This follows from scaling properties, due to the fact that if for some point $z \in B_1$ and $m < \kappa$, we have $c_1 r^m \leq \sup_{B_r(z)} |u(x)| \leq c_2 r^m$, then

$$\Delta u_r(x) = r^{2-m(2-q)} f(rx+z, u_r(x)), \quad u_r(x) := \frac{u(rx+z)}{r^m},$$

and hence u_r tends to a harmonic function (along a subsequence). In particular, analysis close to such points fall under the theory of asymptotically harmonic functions, already studied in [8], in relation to the Thomas–Fermi Atomic model. See also [1,2,10].

In light of Remark 1.1, we shall only be concerned with points of highest order decay for u. However, one may generally divide the set of free boundary points into subsets (see [8,1])

$$\Gamma_m := \{z \in \Gamma : \ c_1 r^m \leq \sup_{B_r(z)} \mid u(x) \mid \leq c_2 r^m, \text{ for some } c_1, c_2 > 0\}, \tag{3}$$

where c_1, c_2 depend on the point z, and $m \leq \kappa$ is either an integer or $m = \kappa$ (observe that κ may also be integer). We shall mainly be interested in the set Γ_κ, which (in some cases) might be of Hausdorff dimension strictly less than $(n-1)$, see Section 3.

[1] It is noteworthy that when $u \geq 0$ then the solution u behave as $u(x) \approx d^\kappa(x, \{u = 0\})$, $(\kappa = 2/(2-q))$ which is the optimal regularity as well as non-degeneracy one can obtain for solutions, (see [3]).