Nonlinear Analysis 151 (2017) 164–186

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Semiclassical states for fractional Schrödinger equations with critical growth

Qing Guo, Xiaoming He*

College of Science, Minzu University of China, Beijing 100081, China

ARTICLE INFO

Article history:
Received 18 June 2016
Accepted 7 December 2016
Communicated by S. Carl

MSC:
35J20
35J65

Keywords:
Fractional Schrödinger equations
Semiclassical solutions
Variational methods

ABSTRACT

In this paper, we consider the following fractional nonlinear Schrödinger equations

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = P(x)g(u) + Q(x)|u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N$$

and prove the existence and concentration of positive solutions under suitable assumptions on the potentials $V(x), P(x)$ and $Q(x)$. We show that the semiclassical solutions u_ε with maximum points x_ε concentrating at a special set $\mathcal{S}_P$ characterized by $V(x), P(x)$ and $Q(x)$. Moreover, for any sequence $x_\varepsilon \to x_0 \in \mathcal{S}_P$, $v_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_\varepsilon)$ convergence strongly in $H^s(\mathbb{R}^N)$ to a ground state solution v of

$$(-\Delta)^s v + V(x_0)v = P(x_0)g(v) + Q(x_0)|v|^{2_s^*-2}v, \quad x \in \mathbb{R}^N.$$

1. Introduction and main results

In this paper, we are concerned with the following fractional nonlinear Schrödinger equations with critical exponents in $\mathbb{R}^N$,

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = P(x)g(u) + Q(x)|u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, $2_s^* := \frac{2N}{N-2s}$, ε is a small parameter, $V(x), P(x)$ and $Q(x)$ are three real continuous function on $\mathbb{R}^N$, and $(-\Delta)^s$ stands for the usual fractional Laplacian. Note that 2_s^* is the critical exponent for Sobolev embedding in $\mathbb{R}^N$, so the nonlinearity is of critical growth at infinity.

The fractional Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\psi + V(x)\psi - f(x,\psi) \tag{1.2}$$

* Corresponding author.
E-mail addresses: guoqing@amss.ac.cn (Q. Guo), xmhe923@muc.edu.cn (X. He).

http://dx.doi.org/10.1016/j.na.2016.12.004

was introduced by Laskin [29,28] through expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, where $(x,t) \in \mathbb{R}^N \times (0,\infty), 0 < s < 1$, and $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function. Similar to the case $s = 1$, standing wave solutions to this equation are solutions of the form $\psi(x,t) = e^{-\frac{i\omega t}{\varepsilon}}u(x)$, where f satisfies that $f(x, e^{i\theta}\psi) = e^{i\theta}f(x,\psi)$ and u solves the elliptic equation

$$\varepsilon^{2s}(-\Delta)^s u + (V(x) - \omega)u = f(x,u). \tag{1.3}$$

Recently, the fractional Schrödinger equation with a nonlinear source term (1.3) is of much interest, and attracts much attention in nonlinear analysis. The existence, uniqueness, regularity results and asymptotic decay properties can be found in, for example, [7,23,21,24,22,34,9,33,26]. For the results of fractional elliptic equations in bounded domain, we refer to [8,35,36,31,32] and the references therein.

A solution ψ is referred to as a bound state of (1.3) if $\psi \to 0$ as $|x| \to \infty$. When ε is sufficiently small, bound states of (1.3) are called semiclassical states, and an important feature of semiclassical states is their concentration as $\varepsilon \to 0$. Concerning the concentration phenomenon, there is a broad literature in the classical case $s = 1$, see for example [3,2,20,25,14,17,15,16,18] and the references therein. Investigation of the existence of solutions concentrating at certain points to nonlocal Schrödinger equations under different conditions have appeared in [13,12,34,11,10,30]. In [13], the authors considered the superlinear problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = u^p, \quad x \in \mathbb{R}^N$$

and obtained the multi-peak solutions via a Lyapunov–Schmidt variational reduction. In [11], a nonlocal problem involving critical or almost critical exponents were considered, see also [10]. In [30], the authors studied the problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = W(x)(f(u) + u^{2_s^*-1}), \quad x \in \mathbb{R}^N$$

and construct two sets dependent on the potential functions as

$$\mathcal{A}_v := \{x \in \mathcal{V} : W(x) = W(x_v)\} \cup \{x \in \mathcal{V} : W(x) > W(x_v)\}$$

and

$$\mathcal{A}_w := \{x \in \mathcal{W} : V(x) = V(x_w)\} \cup \{x \in \mathcal{W} : V(x) < V(x_w)\},$$

where $\mathcal{V} = \{x \in \mathbb{R}^N : V(x) = \min V\}$, $\mathcal{W} = \{x \in \mathbb{R}^N : W(x) = \max W\}$, $W(x_v) = \max_{x\in\mathcal{V}} W(x)$ and $V(x_w) = \max_{x\in\mathcal{W}} V(x)$. They explored the existence, convergence and decay estimate of semiclassical solutions which will concentrate at $\mathcal{A}_v$ or $\mathcal{A}_w$ under two kinds of different assumptions. The structure assumptions like $\mathcal{A}_v, \mathcal{A}_w$ are firstly introduced by Ding and Liu [19] to deal with the existence and concentration of semiclassical states for Schrödinger equations with magnetic fields. Recently, Alves, Xu and Yang [1] employed this kind of assumptions to study Schrödinger–Poisson equations.

Motivated by the references mentioned above, in this paper we consider problem (1.1) and focus on studying how the behavior of the three potentials affect the existence and concentration of the semiclassical solutions with the presence of critical exponent term. We note that problem (1.1) involves three different potentials which are more complicated than that of [30]. This makes the concentration sets more complex and we have to overcome many difficulties, as we shall show in the following section.

By changing variables, if w is a solution of Eq. (1.1), then the function $u(x) = w(\varepsilon x)$ satisfies

$$(-\Delta)^s u + V(\varepsilon x)u = P(\varepsilon x)g(u) + Q(\varepsilon x)|u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N. \tag{1.4}$$

We suppose that the nonlinearity $g : \mathbb{R}^+ \to \mathbb{R}$ verifies the following hypotheses:

$$g(0) = 0, \qquad \lim_{s\to 0} \frac{g(s)}{s} = 0. \tag{1.5}$$