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Blow-up phenomena and local well-posedness for a generalized Camassa–Holm equation with cubic nonlinearity

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ABSTRACT

In this paper we consider the Cauchy problem for a generalized Camassa–Holm equation with cubic nonlinearity in Besov spaces. We first establish the local well-posedness of the equation in the Besov space $B^s_{p,r}$ by using the Littlewood–Paley theory. Then, under a sign condition we reach the sign-preserved property and a precise blow-up criterion. Applying this precise criterion we present a blow-up result and the precise blow-up rate for strong solutions to the equation.

1. Introduction

In this paper we consider the Cauchy problem of the following generalized Camassa–Holm equation

$$\begin{cases} u_t - u_{xxt} = (1+\partial_x)(u^2u_{xx} + uu_x^2 - 2u^2u_x), & t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

If we define (see Lemma 2.15)

$$v = (1-\partial_x)u, \quad t > 0,$$

then, Eq. (1.1) can be changed into a transport-like equation:

$$\begin{cases} v_t + u^2v_x = uv^2 - u^2v, & t > 0, \\ v(x,0) = (1-\partial_x)u_0(x). \end{cases} \tag{1.2}$$

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In this paper we study its well-posedness in Besov spaces. Eq. (1.1) has been recently proposed by Novikov in [44]. He showed that Eq. (1.1) is integrable by using a definition of the existence of an infinite hierarchy of quasi-local higher symmetries [44]. It is easy to see that Eq. (1.1) is of the form:

$$(1-\partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}), \tag{1.3}$$

where F is a homogeneous polynomial, which contains many interesting equations, especially some valuable integrable equations. The most celebrated integrable member of the form (1.3) is the well-known Camassa–Holm (CH) equation [4],

$$(1-\partial_x^2)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}.$$

The CH equation can be regarded as a shallow water wave equation [4,19]. It is completely integrable, which has been studied extensively by many authors. Its complete integrability was studied in [4,10,20]. The CH equation also has a bi-Hamiltonian structure [7,29], and admits exact peaked solitons of the form $ce^{|x-ct|}$ with $c > 0$ which are orbitally stable [21]. These peaked solutions also mimic the pattern specific to the waves of greatest height [11,16,46].

The local well-posedness for the Cauchy problem of CH equation in Sobolev spaces and Besov spaces was established in [14,15,22,45]. It was showed that the CH equation has global strong solutions [8,14,15], finite-time blow-up strong solutions [8,13–15], unique global weak solution [51], and one can distinguish between conservative and dissipative (see [2,3]).

The second celebrated integrable member of (1.3) is the Degasperis–Procesi (DP) equation [25]:

$$(1-\partial_x^2)u_t = 4uu_x - 3u_xu_{xx} - uu_{xxx}.$$

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH shallow water equation [26], also, it is integrable with a bi-Hamiltonian structure [24,18].

Similar to the CH equation, the DP equation has traveling wave solutions [38,47]. The Cauchy problem of the DP equation is locally well-posed in certain Sobolev or Besov spaces [30,34,54], it has global strong solutions [41,54,56], the finite-time blow-up solutions [27,28] and global weak solutions [5,27,55,56]. Different from the CH equation, the DP equation has not only peakon solutions [24], periodic Peakon solutions [55], but also shock peakons [42] and the periodic shock waves [28]. Both CH and DP equation admit cusped traveling waves [37,38], this type of weak solutions are known to arise as solutions for the governing equations for waves in a channel or along a sloped beach [9,31], and also as solutions for the governing equations for equatorial ocean waves [12,17,32,33].

Recently, Novikov introduced an integrable Camassa–Holm type equation with cubic nonlinearity, It may be written in the form [44]:

$$(1-\partial_x^2)u_t = 3uu_xu_{xx} + u^2u_{xxx} - 4u^2u_x.$$

Different from the CH and DP equations, which have quadratic nonlinearity, the Novikov equation has cubic nonlinearity. This equation also had been studied by many authors. Indeed, the Novikov equation is integrable with a bi-Hamiltonian structure and admits exact peakon solutions $u(t,x) = \pm\sqrt{c}e^{|x-ct|}$ with $c > 0$ [35]. It is locally well-posed in certain Sobolev spaces and Besov spaces [49,50,52,53]. It has global strong solutions [49], finite-time blow up solutions [53] and global weak solutions [36,48].

In this paper, firstly, applying Littlewood–Paley theory and transport theory, for the initial data in certain Besov spaces of high regularity or of critical regularity, we prove that Eq. (1.2) is locally well posed in the sense of Hadamard, namely, the local solution to Eq. (1.2) exists uniquely and depends continuously on the initial data. Secondly, we prove a blow-up criterion by means of induction. Then, according to the structure of the equation, we can find a sign-preserved property and a precise blow-up criterion. By virtue of the sign-preserved property, we see that the H^1-norm of u is preserved. Hence finally we obtain a blow-up