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Stability of planar traveling fronts in bistable reaction–diffusion systems

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A B S T R A C T

This paper is concerned with the multidimensional stability of planar traveling fronts in bistable reaction–diffusion systems. It is first shown that planar traveling fronts are asymptotically stable under spatially decaying initial perturbations by appealing to the comparison principle and super-subsolution method. In particular, if the perturbations belong to $L^1(\mathbb{R}^{n-1})$ in a certain sense, we obtain a convergence rate like $t^{-\frac{n-1}{2}}$. Then we show that the solution of the Cauchy problem converges to the planar traveling front with rate $t^{-\frac{n+1}{4}}$ for a spatially non-decaying perturbation with the help of semigroup theory. Finally, we prove that there exists a solution oscillating permanently between two planar traveling fronts, which indicates that planar traveling fronts are not always asymptotically stable in multidimensional space under general bounded perturbations.

1. Introduction

In this paper, we study the large time behavior of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}), & \mathbf{x} = (x_1, \dots, x_n), \quad t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} = (x_1, \dots, x_n), \end{cases} \tag{1.1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, \dots, u_m) \in \mathbb{R}^m$, $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$ with $n \geq 2$. In the sequel, we assume that f satisfies the following hypotheses.

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(H1) There exist only two equilibrium $\mathbf{E}^- < \mathbf{E}^+$ of $\mathbf{f}$, and $\mathbf{E}^\pm$ are stable. That is, $\mathbf{f}(\mathbf{E}^\pm) = \mathbf{0}$, $\lambda_\pm := s(\mathbf{f}'(\mathbf{E}^\pm)) < 0$, where $s(\mathbf{A}) := \max\{\mathrm{Re}\lambda | \det(\lambda\mathbf{I} - \mathbf{A}) = 0\}$. We also assume that the matrices $\mathbf{f}'(\mathbf{E}^\pm)$ are irreducible.

(H2) The nonlinearity $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \dots, f_m(\mathbf{u}))$ is defined on an open domain $\Omega \subset \mathbb{R}^n$ and of class $C^{1+\alpha}$ in $\mathbf{u}$. Moreover, $\mathbf{f}$ satisfies the following conditions:

$$\frac{\partial f_i}{\partial u_j} \geq 0 \quad \text{for all } \mathbf{u} \subset [\mathbf{E}^-, \mathbf{E}^+] \subset \Omega \text{ and } 1 \leq i \neq j \leq m.$$

Moreover, there exist nonnegative constants $L_{ij}^\pm$ such that

$$\frac{\partial f_i}{\partial u_j} + L_{ij}^-\{u_i - E_i^-\}^- + L_{ij}^+\{E_i^+ - u_i\}^- \geq 0 \quad \text{for } i \neq j \text{ and } \mathbf{u} \in \left[\widehat{\mathbf{E}}^-, \widehat{\mathbf{E}}^+\right] \subset \Omega,$$

where $\widehat{\mathbf{E}}^- < \mathbf{E}^- < \mathbf{E}^+ < \widehat{\mathbf{E}}^+$ and for any $a \in \mathbb{R}$,

$$\{a\}^- = \begin{cases} 0, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

According to [19], we define a function $\widetilde{\mathbf{f}} = (\widetilde{f}_1, \dots, \widetilde{f}_m)$ as

$$\widetilde{f}_i(\mathbf{u}) = f_i(\mathbf{u}) + \sum_{1 \leq j \leq m, j \neq i} L_{ij}^-\{u_i - E_i^-\}^-(u_j - E_j^-) + \sum_{1 \leq j \leq m, j \neq i} L_{ij}^+\{E_i^+ - u_i\}^+(u_j - E_j^+)$$

for $\mathbf{u} \subset \left[\widehat{\mathbf{E}}^-, \widehat{\mathbf{E}}^+\right]$. By Theorem 2.2 and Corollaries 1 and 2 in [19], the comparison principle follows immediately on $[\mathbf{E}^-, \mathbf{E}^+]$.

Here, we state some notations which will be used in this paper. For two vectors $\mathbf{c} = (c_1, \dots, c_m)$ and $\mathbf{d} = (d_1, \dots, d_m)$, the symbol $\mathbf{c} < \mathbf{d}$ means $c_i < d_i$ for each $i = 1, \dots, m$ and $\mathbf{c} \leq \mathbf{d}$ means $c_i \leq d_i$ for each $i = 1, \dots, m$. The interval $[\mathbf{c}, \mathbf{d}]$ denotes the set of $\mathbf{u} \in \mathbb{R}^m$ with $\mathbf{c} \leq \mathbf{u} \leq \mathbf{d}$. For $\mathbf{c} = (c_1, \dots, c_m)$, we define $|\mathbf{c}| = \sum_{i=1}^m c_i^2$. For any bounded $\mathbf{u} \in C(\mathbb{R}^n, \mathbb{R}^m)$, we define $\|\mathbf{u}\| = \sup_{\mathbf{x} \in \mathbb{R}^n} |\mathbf{u}(\mathbf{x})|$. For simplicity, we use $\mathbf{0}$ and $\mathbf{1}$ to denote column vectors $(0, \dots, 0)$ and $(1, \dots, 1)$, respectively. In addition, Δ, $\Delta_{\mathbf{x}'}$ and $\nabla_{\mathbf{x}'}$ refer to $\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial \xi^2}$, $\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$ and $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$, respectively.

It is known from [18] that the one dimensional problem

$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

admits a planar traveling front

$$\Phi(x + ct) = (\phi_1(x + ct), \dots, \phi_m(x + ct))$$

satisfying

$$\begin{cases} \phi_i'' - c\phi_i' + f_i(\Phi) = 0, \\ \Phi(\pm\infty) := \lim_{\xi \to \pm\infty} \Phi(\xi) = \mathbf{E}^\pm, \\ \phi_i' > 0 \end{cases} \tag{1.2}$$

for $i = 1, \dots, m$, where $\xi = x + ct$. It is evident that such a front is also a solution of (1.1).

Stability is an important topic in the study of traveling fronts of reaction–diffusion equations [18]. Recently, an increasing attention has been paid to the study of multidimensional stability of traveling fronts.