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The moving plane method for singular semilinear elliptic problems

Annamaria Canino, Luigi Montoro, Berardino Sciunzi*

Università della Calabria, Dipartimento di Matematica e Informatica, Pietro Bucci 31B, I-87036 Arcavacata di Rende, Cosenza, Italy

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ABSTRACT

We consider positive solutions to semilinear elliptic problems with singular nonlinearities, under zero Dirichlet boundary condition. We exploit a refined version of the moving plane method to prove symmetry and monotonicity properties of the solutions, under general assumptions on the nonlinearity.

1. introduction

In this paper we study symmetry and monotonicity properties of positive solutions to the problem

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\gamma > 0$, Ω is a bounded smooth domain of $\mathbb{R}^n$ and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Starting from the pioneering work [14] singular semilinear elliptic equations have been intensely studied, see e.g. [4,5,7,9,10,15,19,21,22,20,26,27]. Furthermore, by a simple change of variables, it also follows that the problem is related to equations involving a first order term of the type $\frac{|\nabla u|^2}{u}$. We refer the readers to [2,6,16] for related results in this setting.

The main difficulties that we have to face are given by the fact that solutions in general are not in $H_0^1(\Omega)$ and the nonlinearity $\frac{1}{s^\gamma} + f(x,s)$ is not Lipschitz continuous at zero. Note that solutions are not in $H_0^1(\Omega)$

* Corresponding author.
E-mail addresses: canino@mat.unical.it (A. Canino), montoro@mat.unical.it (L. Montoro), sciunzi@mat.unical.it (B. Sciunzi).

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already in the case $f \equiv 0$, see [22]. Therefore, in particular, problem (1.1) has to be understood in the weak distributional meaning with test functions with compact support in Ω, that is

$$\int_\Omega (\nabla u, \nabla \varphi)\, dx = \int_\Omega \frac{\varphi}{u^\gamma}\, dx + \int_\Omega f(x,u)\varphi\, dx \quad \forall \varphi \in C^1_c(\Omega). \tag{1.2}$$

The aim of this paper is to prove symmetry and monotonicity properties of the solution under general assumptions on the nonlinearity. Namely we shall consider the case of nonlinearities that fulfill

(hp) $f(x,t)$ is a Carathéodory function which is uniformly locally Lipschitz continuous with respect to the second variable. Namely, for any $M > 0$ given, it follows

$$|f(x,t_1) - f(x,t_2)| \le L_f(M)|t_1 - t_2|, \quad x \in \Omega,\ t_1, t_2 \in [0,\, M].$$
$$|f(x,t)| \le K_f(M), \quad x \in \Omega,\ t \in [0,\, M].$$

The proof is based on the moving plane technique, see [17,25], as developed and improved in [3]. The crucial point here is the lack of regularity of the solutions near the boundary, that is an obstruction to the use of the test functions technique exploited in [3,17,25].

Our main result is the following

Theorem 1.1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (1.1). Assume that the domain Ω is smooth, convex w.r.t. the ν-direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

With the notation $x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x\cdot\nu)\nu$, assume that f satisfies (hp), $f(\cdot,t)$ is non decreasing in the $x\cdot\nu$-direction in the set $\Omega_0^\nu := \Omega \cap \{x\cdot\nu < 0\}$, for all $t \in [0,\infty)$ and

$$f(x,t) = f(x_0^\nu, t) \quad \text{if } x \in \Omega_0 \text{ and } t \in [0,\infty).$$

Then u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the ν-direction in Ω_0^ν. In particular, if Ω is a ball centered at the origin of radius $R > 0$, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $0 < r < R$.

Note that the monotonicity assumption on f, with respect to the first variable, is necessary for the applicability of the moving plane method and for the validity of the result. This is well known already in the case of non singular nonlinearities. Our theorem recovers and improves the previous result in [10] (see also the applications in [7,8,11]) where a monotonicity assumption on the second variable of $f(x,\cdot)$ on the nonlinearity was required. In fact such a condition was necessary in [10] to use the decomposition of the solution

$$u = u_0 + w \quad \text{for some } w \in H_0^1(\Omega)$$

provided in [9] where u_0 is the solution to the pure singular problem: $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ and

$$\begin{cases} -\Delta u_0 = \dfrac{1}{{u_0}^\gamma} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

See [9,12,13,23] for the uniqueness of the solution and [5,9] for the existence of the solution.

We develop a new technique that allows us to avoid the use of such a decomposition and this is the key point in order to obtain the full symmetry result, namely to consider the general case of locally Lipschitz continuous nonlinearities. A crucial point in the proof is the study of the problem near the boundary. We combine a fine analysis of the behavior of the solution near the boundary based on comparison arguments that go back to [9] with some ideas from [24] and with an improved test functions technique.