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Isolated singularities of solutions of defocusing Hartree equation

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ABSTRACT

Our purpose of this paper is to study isolated singular positive solution u of defocusing Hartree equation

$$\begin{cases}-\Delta u + u + I_\alpha[u^p]u^q = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},\\ \lim\limits_{|x|\to+\infty} u(x) = 0,\end{cases} \tag{1}$$

where p, $q > 0$, $N \geq 3$, $\alpha \in (0, N)$ and

$$I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}}\, dy.$$

We obtain that the solution $u \in L^1_{loc}(\mathbb{R}^N)$ of (1) satisfying that $u^p \in L^1(\mathbb{R}^N)$, $I_\alpha[u^p]u^q \in L^1_{loc}(\mathbb{R}^N)$, is a weak solution of

$$\begin{cases}-\Delta u + u + I_\alpha[u^p]u^q = k\delta_0 \quad \text{in } \mathbb{R}^N,\\ \lim\limits_{|x|\to+\infty} u(x) = 0.\end{cases} \tag{2}$$

Furthermore, the classical solution of (1) is derived by considering the very weak solution of (2). To this end, we make use of Schauder fixed point theorem to obtain the existence of weak solutions of (2).

1. Introduction

In this paper, we are concerned with the existence of singular positive solutions to defocusing Hartree equation

$$\begin{cases}-\Delta u + u + I_\alpha[u^p]u^q = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},\\ \lim\limits_{|x|\to+\infty} u(x) = 0,\end{cases} \tag{1.1}$$

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where p, $q > 0, N \geq 3, \alpha \in (0, N)$ and

$$I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}}\, dy.$$

The nonlinearity $I_\alpha[u^p]u^q$ involves the Riesz potential I_α with $\alpha \in (0, N)$, which can be interpreted as the inverse of $(-\Delta)^{\frac{\alpha}{2}}$.

Problem (1.1) concerns with defocusing Hartree term, in contrast with the problem with focusing Hartree nonlinearity, which is also known as Choquard or Schrödinger–Newton equation, that is semilinear elliptic equation

$$-\Delta u + u = I_\alpha[u^p]u^q. \tag{1.2}$$

The focusing Hartree equation is used as an approximation to the Hartree–Fock theory by Lieb et al. in [12,13] and has been advancing the theory of nonlinear partial differential equations, see [1,16,17,20], the survey [15] and the references therein. Meanwhile the Cauchy problem involving the defocusing Hartree term has been deeply studied, in the aspects of the global well-posedness and long range scattering theory, from the nonlocal properties of the defocusing Hartree term [10,11,21,22].

The signs of nonlinearities play an essential role in the study of isolated singularities for semilinear elliptic problems. One important type of nonlinearity is the source. The typical model is

$$\begin{cases} -\Delta u = u^p \quad \text{in } \Omega \setminus \{0\}, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

whose positive singularities were first classified by Lions in [14], showing the connection of singular solutions of (1.3) with very weak solutions of the equation

$$\begin{cases} -\Delta u = u^p + k\delta_0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain containing the origin and δ_0 is Dirac mass at the origin. Later on, by using dynamic analysis, the singularities of (1.3) have been studied by Aviles in [2] for $p = \frac{N}{N-2}$, by Gidas–Spruck in [10] for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and by Caffarelli–Gidas–Spruck in [4] for $p = \frac{N+2}{N-2}$.

The second type of nonlinearity is the absorption. The elliptic equation with absorption

$$\begin{cases} -\Delta u + u^q = 0 \quad \text{in } \Omega \setminus \{0\}, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.5}$$

has been studied by Brezis and Véron in [3], where it is proved that problem (1.5) admits only zero solution when $q \geq N/(N-2)$. When $1 < q < N/(N-2)$, Véron in [18] described all the possible singular behaviors of positive solutions of (1.5):

(i) either $u(x) \sim c_N k|x|^{2-N}$ as $x \to 0$ and k can take any positive value; in this case u is said to have a *weak singularity* at 0, denoted by u_k, and u_k is the unique weak solution of

$$\begin{cases} -\Delta u + u^q = k\delta_0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega; \end{cases} \tag{1.6}$$

(ii) or $u(x) \sim c_{N,q}|x|^{-\frac{2}{q-1}}$ as $x \to 0$; in this case u is said to have a *strong singularity* at 0, and $u = \lim_{k\to\infty} u_k$.

This classification is based on the Comparison Principle for $Lu = -\Delta u + u^q$, and it has been extended to the case of fractional semilinear elliptic equations, see the Refs. [5–7].

In the subcritical case, i.e. $p < \frac{N}{N-2}$, the isolated singularities of (1.3) and (1.5) are solved by building the connection with the very weak solutions of corresponding equations involving Dirac mass in both types