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Global L^∞-estimates and Hölder continuity of weak solutions to elliptic equations with the general nonstandard growth conditions

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ABSTRACT

We study the elliptic equations in divergence form with the general nonstandard growth conditions involving Lebesgue measurable functions on Ω. In this paper we prove global L^∞-estimates and Hölder continuity of weak solutions for these equations. Our results improve the one in Giusti (2003) even for the case when $p(x) \equiv p$ and $q(x) \equiv q$ are constants.

1. Introduction and main results

In this paper we deal with the regularity properties of weak solutions to the quasilinear elliptic equation

$$-\operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u) = 0 \quad \text{in}\,\Omega, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n, n \geq 2$ and assume that the coefficients $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions satisfying the following structural conditions:

$$A(x,z,\xi)\xi \geq a_0|\xi|^{p(x)} - a_1(x)|z|^{q(x)} - a_2(x), \tag{1.2}$$

$$|A(x,z,\xi)| \leq b_0(x)|\xi|^{p(x)-1} + b_1(x)|z|^{q(x)-1} + b_2(x), \tag{1.3}$$

$$|B(x,z,\xi)| \leq c_0(x)|\xi|^{\frac{p(x)}{q'(x)}} + c_1(x)|z|^{q(x)-1} + c_2(x) \tag{1.4}$$

for a.e. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, where p and q are the functions such that

$$p \in C(\bar{\Omega}), \qquad q \in C(\bar{\Omega}), \tag{1.5}$$

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$$1 < p^- := \inf_{\Omega} p(x) \le p^+ := \sup_{\Omega} p(x) < \infty, \tag{1.6}$$

$$p(x) \le q(x) < p^*(x) := \begin{cases} \dfrac{np(x)}{n-p(x)}, & \text{if } p(x) < n, \\ +\infty, & \text{if } p(x) \ge n \end{cases} \tag{1.7}$$

and a_0 is a positive constant and $a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are certain non-negative Lebesgue measurable functions and $q'(x) = \frac{q(x)}{q(x)-1}$.

The prototype of these equations is $p(x)$-Laplace equation

$$-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = 0.$$

In recent years increasing attention has been paid to the study of the differential equations with non-standard growth conditions that arise from nonlinear elasticity, electro-rheological fluids and non-Newtonian fluids; see, e.g. [2,27,28,31]. There are some essential differences between the variable exponent problems and the constant exponent problems. The operator $-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian which is a generalization of the p-Laplacian and possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous.

We can see that the inhomogeneity due to variable exponent is a main difficulty in generalizing the results in the constant exponent problem to the variable exponent one and it is a source of singular phenomena in the variable exponent problems (see, e.g. [13,22,25,26]).

The regularity results of the weak solutions to (1.1) with the structural conditions (1.2)–(1.4) when $p(\cdot) \equiv p$ and $q(\cdot) \equiv q$ (p, q-constants) have been studied sufficiently; see, e.g. [7,18,23].

Recently, the boundedness and Hölder continuity of weak solutions to elliptic equations with nonstandard growth have been studied by several authors; see, e.g. [3–5,21]. Also, we mention that the first $C^{1,\alpha}$-regularity results for solutions to the $p(x)$-Laplacian equations have been obtained in [1,6]. On the other hand, some researchers have studied the interesting other results for the variable exponent elliptic equations: existence, uniqueness and $C^{1,\alpha}$ regularity; see, e.g. [12,10,30]. Also see the overview paper [20].

The aim of the present paper is to prove the global L^∞-estimates and Hölder continuity of the weak solutions of (1.1) with the conditions (1.2)–(1.7).

The weak solutions related to the general structure conditions (1.2)–(1.7) have been considered in [14], where local boundedness and Hölder continuity are proven in the case that $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1$ and c_2 are positive constants. In [15], a similar problem has been studied (and establishes similar results) but when a_2, b_2 and c_2 are certain non-negative Lebesgue functions.

In [24], Lukkari has been studied Hölder continuity of weak solutions up to the boundary in domains with uniformly fat complements, provided of course that the boundary values are Hölder continuous but his assumptions on structure conditions are slightly stronger than [14] (see also [9]).

To our knowledge, the present paper seems to be the first to study the problem with the most general structure conditions. Our topic is to find the optimal assumptions on $a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ so as to involve the corresponding results in the constant exponent case.

We will use the variable exponent spaces $L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega), W_0^{1,p(\cdot)}(\Omega)$, the definitions of which will be given in Section 2. The symbols of some common spaces used in this paper such as $L^\infty(\Omega), C^1(\bar{\Omega})$, $C(\bar{\Omega}), C^\alpha(\bar{\Omega})$ and $C^\alpha(\partial\Omega)$, are standard. Now, we will state the main results of this paper. Our first result is,

Theorem 1.1. *Let (1.2)–(1.7) be satisfied, where a_0 is a positive constant and $a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are non-negative measurable functions satisfying*

$$a_1, b_1, c_1 \in L^{r(\cdot)}(\Omega), \qquad a_2 \in L^{s(\cdot)}(\Omega), \qquad b_2, c_2 \in L^{h'(\cdot)s(\cdot)}(\Omega), \qquad b_0 \in L^{p(\cdot)s(\cdot)}(\Omega), \qquad c_0 \in L^{q(\cdot)r(\cdot)}(\Omega), \tag{1.8}$$